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Illustration of Sampling-Based Methods for Uncertainty and Sensitivity Analysis

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Abstract

A sequence of linear, monotonic, and nonmonotonic test problems is used to illustrate sampling-based uncertainty and sensitivity analysis procedures. Uncertainty results obtained with replicated random and Latin hypercube samples are compared, with the Latin hypercube samples tending to produce more stable results than the random samples. Sensitivity results obtained with the following procedures and/or measures are illustrated and compared: correlation coefficients (CCs), rank correlation coefficients (RCCs), common means (CMNs), common locations (CLs), common medians (CMDs), statistical independence (SI), standardized regression coefficients (SRCs), partial correlation coefficients (PCCs), standardized rank regression coefficients (SRRCs), partial rank correlation coefficients (PRCCs), stepwise regression analysis with raw and rank-transformed data, and examination of scatterplots. The effectiveness of a given procedure and/or measure depends on the characteristics of the individual test problems, with (i) linear measures (i.e., CCs, PCCs, SRCs) performing well on the linear test problems, (ii) measures based on rank transforms (i.e., RCCs, PRCCs, SRRCs) performing well on the monotonic test problems, and (iii) measures predicated on searches for nonrandom patterns (i.e., CMNs, CLs, CMDs, SI) performing well on the nonmonotonic test problems.

Key Words: Chi-square, common mean, common median, correlation coefficient, epistemic uncertainty, Kruskal-Wallis, Latin hypercube sampling, Monte Carlo, partial correlation coefficient, random sampling, rank transform, regression analysis, replicated sampling, scatterplot, sensitivity analysis, standardized regression coefficient, statistical independence, stepwise regression, subjective uncertainty, uncertainty analysis.

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1. Introduction

Sampling-based methods for uncertainty and sensitivity analysis have become very popular (e.g., Refs. [1-13]). Such methods involve the generation and exploration of a mapping from uncertain analysis inputs to uncertain analysis results (e.g., Refs. [14-23]).

The analysis or model under consideration can be represented by a vector function

$$\mathbf{y} = \mathbf{y}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \tag{1}$$

where

$$\mathbf{x} = [x_1, x_2, \ldots, x_{nX}] \tag{2}$$

and

$$\mathbf{y} = [y_1, y_2, \ldots, y_{nY}] \tag{3}$$

designate the inputs to the analysis and the outcomes of the analysis, respectively. In real analyses, the dimensions nX and nY of $\mathbf{x}$ and $\mathbf{y}$ can be large (e.g., >100). Further, the function $\mathbf{f}$ can be quite complex (e.g., a model that involves the numerical solution of a system of nonlinear partial differential equations or possibly a probabilistic risk assessment for a complex engineered facility such as a nuclear power plant).

If the value for $\mathbf{x}$ was unambiguously known, then $\mathbf{y}(\mathbf{x})$ could be determined and presented as the unique outcome of the analysis. However, there is uncertainty with respect to the appropriate value to use for $\mathbf{x}$ in most analyses, with the result that there is also uncertainty in the value of $\mathbf{y}(\mathbf{x})$. The uncertainty in $\mathbf{x}$ and its associated effect on $\mathbf{y}(\mathbf{x})$ lead to two closely related questions: (i) "What is the uncertainty in $\mathbf{y}(\mathbf{x})$ given the uncertainty in $\mathbf{x}$?", and (ii) "How important are the individual elements of $\mathbf{x}$ with respect to the uncertainty in $\mathbf{y}(\mathbf{x})$?" Attempts to answer these two questions are typically referred to as uncertainty analysis and sensitivity analysis, respectively.

An assessment of the uncertainty in $\mathbf{y}$ derives from a corresponding assessment of the uncertainty in $\mathbf{x}$. In particular, $\mathbf{y}$ is assumed to have been developed so that appropriate analysis results are obtained if the appropriate value for $\mathbf{x}$ is used in the evaluation of $\mathbf{y}$. Unfortunately, it is impossible to unambiguously specify the appropriate value of $\mathbf{x}$ in most analyses; rather, there are many possible values for $\mathbf{x}$ of varying levels of plausibility. Such uncertainty is often given the designation subjective or epistemic (e.g., Refs. [24-31]) and is characterized by assigning a distribution

$$D_1, D_2, \ldots, D_{nX} \tag{4}$$

to each element x_i of $\mathbf{x}$. Correlations and other restrictions involving the x_i are also possible. These distributions and any associated conditions characterize a degree of belief as to where appropriate value of each variable x_i is located for use in evaluation of $\mathbf{y}$ and in turn lead to distributions for the individual elements of $\mathbf{y}$. Given that the distributions in Eq. (4) characterize a degree of belief with respect to where the appropriate input to use in the analysis is located, the resultant distributions for the elements of $\mathbf{y}$ characterize a corresponding degree of belief with respect to where the appropriate values of the outcomes of the analysis are located. The distributions in Eq. (4) are often developed through an expert review process (e.g., Refs. [32-53]).

Sampling-based methods for uncertainty and sensitivity analysis are based on a sample

$$\mathbf{x}_k = \left[x_{k1}, x_{k2}, \ldots, x_{k,nX}\right],\ k = 1, 2, \ldots, nS, \tag{5}$$

of size nS from the possible values for $\mathbf{x}$ as characterized by the distributions in Eq. (4) and on the corresponding evaluations

$$\mathbf{y}(\mathbf{x}_k) = \left[y_1(\mathbf{x}_k),\ y_2(\mathbf{x}_k), \ldots, y_{nY}(\mathbf{x}_k)\right],\ k = 1, 2, \ldots, nS, \tag{6}$$

of $\mathbf{y}$. The pairs

$$\left[\mathbf{x}_k, \mathbf{y}(\mathbf{x}_k)\right],\ k = 1, 2, \ldots, nS, \tag{7}$$

form a mapping from the uncertain analysis inputs (i.e., the $\mathbf{x}_k$'s) to the corresponding uncertain analysis results (i.e., the $\mathbf{y}(\mathbf{x}_k)$'s).

When an appropriate probabilistic procedure, such as random sampling or Latin hypercube sampling,[54] has been used to generate the sample in Eq. (5) from the distributions in Eq. (4), the resultant distributions for the elements of $\mathbf{y}$ characterize the uncertainty in the results of the analysis (i.e., constitute the outcomes of an uncertainty analysis). Further, examination of scatterplots, regression analysis, partial correlation analysis, and other procedures for investigation the mapping in Eq. (7) provide a way to determine the effects of the elements of $\mathbf{x}$ on the elements of $\mathbf{y}$ (i.e., constitute procedures for sensitivity analysis).

The purpose of this presentation is to use selected test problems from a recent book on sensitivity analysis[55] to illustrate sampling-based methods for uncertainty and sensitivity analysis. No attempt is made to present results for all test problems. Rather, the problems to be discussed were selected because they either provided representative results or interesting analysis challenges. All the problems involve a single analysis outcome and thus have the form

$$y = f(\mathbf{x}) \tag{8}$$

rather than the more common and complex vector form in Eq. (1).

To illustrate the effects of sampling procedures, each problem is evaluated with 10 independent Latin hypercube samples (LHSs) of size 100 each and also 10 independent random samples of size 100 each. Sensitivity analysis results will be presented for 1 LHS of size 100 (i.e., $nLHS$ = 100); in some instances, sensitivity analysis results will also be presented for the 1000 sample elements that result from pooling the 10 LHSs (i.e., $nLHS$ = 1000). The sensitivity analysis procedures and/or measures considered will include correlation coefficients (CCs), rank correlation coefficients (RCCs), common means (CMNs), common locations (CLs), common medians (CMDs), statistical independence (SI), standardized regression coefficients (SRCs), partial correlation coefficients (PCCs), standardized rank regression coefficients (SRRCs), partial rank correlation coefficients (PRCCs), stepwise regression analysis with raw and rank-transformed data, and examination of scatterplots.

It is hoped that the presentation of these results will help the reader develop insights with respect to the behavior and effectiveness of the techniques under consideration. Use of these relatively simple test problems has several advantages over the use of more complex problems, including (i) clear specification of the model under consideration, (ii) low computational cost that allows the consideration of replicated samples and samples of different sizes, and (iii) the opportunity for independent solution of the problems by the interested reader.

The presentation is organized as follows. The uncertainty and sensitivity analysis procedures in use are briefly described (Sect. 2). Then, example analyses involving linear test problems (Sect. 3), monotonic test problems (Sect. 4), and nonmonotonic test problems (Sect. 5) are presented. Finally, the presentation ends with a concluding discussion (Sect. 5).

2. Uncertainty and Sensitivity Analysis Procedures

This presentation will illustrate the use of both random sampling and Latin hypercube sampling in the generation of the mapping between analysis inputs and analysis results in Eqs. (5)–(7). For convenience in distinguishing between sampling procedures in subsequent discussions, random and LHSs will be referred to as being of size nR and $nLHS$, respectively, rather than of size nS as in Eqs. (5)–(7).

In the absence of correlations between the elements of $\mathbf{x}$, random sampling operates in the following manner to generate the sample in Eq. (5). To produce $\mathbf{x}_k$, each element x_{ki}, $i = 1, 2, \ldots, nX$, of $\mathbf{x}_k$ is randomly selected from its distribution D_i in Eq. (4). This sampling is carried out independently for each $\mathbf{x}_k$ to produce the nR (i.e., nS) sample elements in Eq. (5).

Unlike random sampling, Latin hypercube sampling is a stratified sampling technique, and the sample elements $\mathbf{x}_k$ in Eq. (5) cannot be generated independently of each other. In the absence of correlations between the elements of $\mathbf{x}$, Latin hypercube sampling operates in the following manner to generate the sample in Eq. (5). The range of each variable (i.e., the x_i, $i = 1, 2, \ldots, nX$) is divided into $nLHS$ intervals of equal probability and one value is selected at random from each interval in consistency with the distributions in Eq. (4). The *nLHS* values thus obtained for x_1 are paired at random and without replacement with the *nLHS* values obtained for x_2. These *nLHS* pairs are combined in a random manner without replacement with the *nLHS* values of x_3 to form *nLHS* triples. This process is continued until a set of *nLHS* *nX*-tuples is formed. This set of *nX*-tuples constitutes an LHS of size *nLHS* (i.e., the sample of size *nS* in the context of Eq. (5)).

Unlike random sampling, Latin hypercube sampling ensures a full stratification over the range of each sampled variable. Additional discussion and illustration of random and Latin hypercube sampling is given in Sect. 6.3 of Ref. [56]. Correlations can be imposed on both random and LHSs with a restricted pairing technique developed by Iman and Conover [57-59]. None of the test problems under consideration involve correlated variables; thus, there was no requirement to induce nonzero correlations between variables. However, the restricted pairing technique was used to assure that correlations between sampled variables were close to zero (see Sect 3.2 of Ref. [19] for an introductory description of this technique).

Both random and Latin hypercube sampling provide a basis for uncertainty analysis. In particular, each sample element can be assigned a weight (i.e., a probability in common but incorrect usage) equal to the reciprocal of the sample size that can be used in the construction probabilistic representations of the uncertainty in analysis outcomes. Possible representations include cumulative distribution functions (CDFs), complementary cumulative distribution functions (CCDFs), box plots, and means and standard deviations (Sect. 6.5, Ref. [56]). This presentation will use CDFs to display the uncertainty in the outcomes of the test problems.

The simplest procedure for exploring the mapping in Eq. (7) is the examination of scatterplots, which are plots with the values of a sampled variable on one axis and the corresponding values of the analysis outcome on the other axis. Specifically, a scatterplot is simply a plot of the points

$$\left(x_{ki}, y_k\right),\ k = 1, 2, \ldots, nS, \tag{9}$$

for the uncertain variable x_i in the sample in Eq. (5).

A simple but formal method to assess the relationship between analysis input and analysis results is to calculate CCs between sampled variables and corresponding analysis outcomes. For the sequence of observations in Eq. (9), the (sample or Pearson) correlation $r_{x_i y}$ between x_i and y is defined by

$$r_{x_i y} = \frac{\sum_{k=1}^{nS} (x_{ki} - \bar{x}_i)(y_k - \bar{y})}{\left[\sum_{k=1}^{nS} (x_{ki} - \bar{x}_i)^2\right]^{1/2} \left[\sum_{k=1}^{nS} (y_k - \bar{y})^2\right]^{1/2}}, \tag{10}$$

where

$$\overline{y} = \sum_{k=1}^{nS} y_k / nS, \ \ \overline{x}_i = \sum_{k=1}^{nS} x_{ki} / nS \ .$$

The CC $r_{x_i y}$ takes values between –1 and 1 and provides a measure of the linear relationship between x_i and y. The quantity $z = r_{x_i y}\sqrt{nS}$ is distributed approximately normally with mean 0 and standard deviation 1 when x_i and y are uncorrelated, x_i and y have enough convergent moments (i.e., the tails of their distributions die off sufficiently rapidly), and nS is sufficiently large (p. 631, Ref. [60]), and thus can be used to test for the significance of $r_{x_i y}$ (i.e., to determine the probability, or p-value, that a CC $\tilde{r}_{x_i y}$ satisfying $\left|\tilde{r}_{x_i y}\right| \geq \left|r_{x_i y}\right|$ would occur by chance in the presence of no relationship between x_i and y; see Eqs. (6.6.38) – (6.6.40) in Ref. [56] for additional discussion).

The CC $r_{x_i y}$ measures the effect of one variable (i.e., x_i) at a time on y. Regression analysis can be used to assess the combined effects of multiple variables on y. Specifically, least squares procedures can be used to construct the regression model

$$\hat{y} = b_0 + \sum_{i=1}^{nX} b_i x_i , \tag{11}$$

where b_0, b_1, …, b_{nX} are coefficients determined in the construction of the regression model ([61-65]; Sect. 6.6.2, Ref. [56]). The signs of the coefficients b_1, b_2, …, b_{nX} indicate whether y increases (i.e., a positive coefficient) or decreases (i.e., a negative coefficient) as the corresponding x value increases. Further, the regression model in Eq. (11) has associated with it a quantity called an R^2 value, or coefficient of determination, that is equal to the fraction of the uncertainty in y that can be accounted for by the regression model (see Eqs. (6.6.11) and (6.6.14), Ref. [56]). When the variables $x_1, x_2, \ldots, x_{nX}$ are independent,

$$R^2 = R_1^2 + R_2^2 + \cdots + R_{nX}^2 , \tag{12}$$

where $R_i^2, i = 1,2,\ldots, nX$, is the R^2 value that results from regressing y on only x_i.

The usefulness of the coefficients b_1, b_2, …, b_{nX} in Eq. (11) in sensitivity analysis is severely limited by the fact that they depend on the units in which y and the x_i are expressed. Because of this, the regression model in Eq. (11) is usually expressed in the following normalized form:

$$\left(\hat{y} - \overline{y}\right)/\hat{s} = \sum_{i=1}^{nX} \left(b_i \hat{s}_i / \hat{s}\right)\left(x_i - \overline{x}_i\right)/\hat{s}_i , \tag{13}$$

where

$$\hat{s} = \left[\sum_{k=1}^{nS} \left(y_k - \overline{y} \right)^2 \Big/ \left(nS - 1 \right) \right]^{1/2}, \quad \hat{s}_i = \left[\sum_{k=1}^{nS} \left(x_{ki} - \overline{x}_i \right)^2 \Big/ \left(nS - 1 \right) \right]^{1/2},$$

and $\overline{y}$ and $\overline{x}_i$ are defined in conjunction with Eq. (10). The coefficients $b_i \hat{s}_i / \hat{s}$ appearing in Eq. (13) are called standardized regression coefficients (SRCs). When the x_i are independent, the absolute value of the SRCs can be used to provide a measure of variable importance. Specifically, the coefficients provide a measure of importance based on the effect of moving each variable away from its expected value by a fixed fraction of its standard deviation while retaining all other variables at their expected values. Calculating SRCs is equivalent to performing the regression analysis with the input and output variables normalized to mean zero and standard deviation one.

Determination of the regression coefficients in Eq. (11) and the SRCs in Eq. (13) is based entirely on procedures involving minimization of functions and algebraic manipulations and entails no statistics. If desired, formal statistical procedures can be used to indicate if these coefficients appear to be different from zero (Sect. 6.6.3, Ref. [56]). In particular, these procedures provide the probability (i.e., the p-value) that a stronger linear relationship would appear by chance alone if there was no relationship between the variables involved. However, such procedures are based on assumptions that are not satisfied in sampling-based sensitivity studies of deterministic models (i.e., models for which a given input always produces the same result), and thus the outcome of using formal statistical procedures to make assessments about the significance of individual coefficients or other entities in sampling-based sensitivity studies should be regarded simply as one form of guidance as to whether or not a model prediction appears to be affected by a particular model input.

When many uncertain variables are under consideration (i.e., when nX is large), construction and presentation of the regression models in Eqs. (10) and (12) with all nX variables is unwieldy and typically unnecessary. In this situation, the regression models are usually constructed in a stepwise manner in which one variable at a time is added to the regression model until a point is reached at which no additional significant variables can be identified (i.e., a stepwise regression analysis is carried out; see Sect. 6.6.5, Ref. [56]). Variable importance is indicated by the order in which the variables enter the regression model, the size of the SRCs for the individual variables, and the changes in R^2 values as successive variables enter the regression model. A specified significance level (i.e., p-value) is usually used to define a stopping point for the stepwise procedure.

The CC $r_{x_i y}$ is perhaps best interpreted in the context of regression analysis. Specifically, the following regression model relating x and y can be constructed with least squares procedures:

$$\hat{y} = b_0 + b_i x_i \,. \tag{14}$$

The definition of $r_{x_i y}$ in Eq. (10) is equivalent to the definition

$$r_{x_i y} = sign(b_i)\left(R_i^2\right)^{1/2}, \tag{15}$$

where $sign(b_i) = 1$ if $b_i \geq 0$, $sign(b_i) = -1$ if $b_i < 0$ and R_i^2 is the coefficient of determination that results from regressing y on x_i. Thus, $r_{x_i y}$ captures both the sign of the regression coefficient b_i and the fraction of the uncertainty in y that can be accounted for by a linear relationship involving x_i; further, if b_i is a SRC, then $r_{x_i y} = b_i$.

The CC $r_{x_i y}$ represents the linear relationship between x_i and y but makes no correction for the possible effects on y of other uncertain variables. The PCC provides a representation of the linear relationship between two variables after a correction has been made to remove the linear effects of all other variables in the analysis (Sect. 8.4, Ref. [56]). The PCC between an individual variable x_i and y is obtained from the use of a sequence of regression models. First, the following two regression models are constructed:

$$\hat{x}_i = c_0 + \sum_{\substack{p=1 \\ p \neq i}}^{nX} c_p x_p \text{ and } \hat{y} = b_0 + \sum_{\substack{p=1 \\ p \neq i}}^{nX} b_p x_p. \tag{16}$$

Then, the results of the two preceding regressions are used to define the new variables $x_i - \hat{x}_i$ and $y - \hat{y}$. The PCC $p_{x_i y}$ between x_i and y is the CC between $x_i - \hat{x}_i$ and $y - \hat{y}$. Thus, the PCC provides a measure of the linear relationship between x_i and y with the linear effects of the other variables removed.

Thus far, CCs, SRCs and PCCs have been introduced as measures of the relationship between uncertain (i.e., sampled) variables and analysis results. These coefficients are based on determining linear relationships and typically perform poorly when the underlying relationships are nonlinear. When these relationships are nonlinear, but still monotonic, the rank transformation can be used to linearize the underlying relationships between sampled and calculated variables ([66, 67]; Sect. 8.6, Ref. [56]). With the rank transformation, data are replaced with their corresponding ranks, and then the usual regression and correlation procedures are performed on these ranks. Specifically, the smallest value of each variable is assigned the rank 1, the next largest value is assigned the rank 2, and so on up to the largest value, which is assigned the rank nS, where nS denotes the number of observations (i.e., samples). Further, averaged ranks are assigned to equal values of a variable. The analysis is then performed with these ranks being used as the values for the input and output variables. The outcomes of such analysis are RCCs, SRRCs and PRCCs instead of CCs, SRCs and PCCs, respectively. In essence, the use of rank-transformed data results in an analysis based on the strength of monotonic relationships rather than on the strength of linear relationships.

When regression-based approaches to sensitivity analysis (i.e., CCs, SRCs, PCCs, RCCs, SRRCs, PRCCs) do not yield satisfactory insights, important variables can be searched for by attempting to identify patterns in the mapping in Eq. (7) with techniques that are not predicated on searches for linear or monotonic relationships. Possibilities include use of (i) the F-statistic to identify changes in the mean value of y across the range of individual x_i's, (ii) the

χ^2-statistic to identify changes in the median value of y across the range of individual x_i's (iii) the Kruskal-Wallis statistic to identify changes in the distribution of y across the range of individual x_j's, and (iv) the χ^2-statistic to identify nonrandom joint distributions involving y and individual x_i's [68, 69]. For convenience, the preceding are referred to as tests for (i) common means (CMNs), (ii) common medians (CMDs), (iii) common locations (CLs), and (iv) statistical independence (SI), respectively.

The preceding statistics are based on dividing the values of x_i in Eq. (9) into intervals. Typically, these intervals contain equal numbers of values for x_i (i.e., the intervals are of equal probability); however, this is not always the case (e.g., when x_i has a finite number of values of unequal probability). The calculation of the F-statistic for CMNs and the Kruskal-Wallis statistic for CLs involves only the division of x_i into intervals. The F-statistic and the Kruskal-Wallis statistic are then used to indicate if the y values associated with these intervals appear to have different means and distributions, respectively. The χ^2-statistic for CMDs involves a further partitioning of the y values into values above and below the median for all y in Eq. (9), with the corresponding significance test used to indicate if the y values associated with the individual intervals defined for x_i appear to have medians that are different from the median for all values of y. The χ^2-statistic for SI involves a partitioning of the y values in Eq. (9) into intervals of equal probability analogous to the partitioning of the values of x_i, with the corresponding significance test used to indicate if the distribution of the points (x_{ki}, y_k) over the resultant cells appears to be different from what would be expected if there was no relationship between x_i and y. For each statistic, a p-value can be calculated which corresponds to the probability of observing a stronger pattern than the one actually observed if there is no relationship between x_i and y. An ordering of p-values then provides a ranking of variable importance (i.e., the smaller the p-value, the stronger the effect of x_i on y appears to be). More detail on these and other related procedures is given in Refs. [68, 69]. Further, the use of tests based on CMNs, CMDs, CLs and SI is extensively illustrated in the analyses for the individual test problems.

3. Linear Test Problems

The first linear test problem (Model 1, Ref. [55]) is defined by

$$f(\mathbf{x}) = \sum_{i=1}^{3} x_i, \quad \mathbf{x} = [x_1, x_2, x_3], \tag{17}$$

with $x_i : U(\bar{x}_i - \sigma_i, \bar{x}_i + \sigma_i)$, $\bar{x}_i = 3^{i-1}$, $\sigma_i = 0.5\,\bar{x}_i$ for i = 1, 2, 3, and x:$U(a, b)$ used to indicate that x has a uniform distribution on $[a, b]$. Thus, the D_i , i =1, 2, 3, in Eq. (4) correspond to uniform distributions in this test problem.

The distributions assigned to the x_i lead to a distribution for $f(\mathbf{x})$, with Latin hypercube sampling tending to produce more stable estimates of this distribution than random sampling (Fig. 1).

For the $nLHS$ = 100, CCs, RCCs, CMNs, CLs, CMDs and SI all identify x_3 as the most important variable; CCs and RCCs also indicate an effect for x_2 (Table I). Due to the large size of the p-values (i.e., > 0.05), CMNs, CLs, CMDs and SI do not indicate an effect for x_2, and none of the tests indicate an effect for x_1.

The division of the x and y values for use in the test for SI is illustrated in Fig. 2. The tests for CMNs and CLs only use the indicated divisions of the x-axis. The test for CMDs uses the indicated divisions of the x-axis and an additional division of the y-axis into values above and below the median.

For $nLHS$ = 1000, CCs, RCCs, CMNs, CLs, CMDs and SI identify x_3 and x_2 as the two most important variables (Table I). Further, CCs, RCCs, CMNs and CLs also indicate an effect for x_1. Thus, as might be expected, the larger sample is leading to more resolution in the sensitivity analysis. However, CCs and RCCs were able to identify the two most important variables with a sample of size 100.

Examination of scatterplots clearly shows the dominant effect of x_3 (Fig. 2). The effect of x_2 is barely discernible in the scatterplot for $nLHS$ = 100 but is easily seen for $nLHS$ = 1000. The scatterplots for x_1 (not shown) indicate no visually discernible effect for $nLHS$ = 100 and a barely discernible effect for $nLHS$ = 1000.

In addition to various tests of significance (Table I) and the examination of scatterplots (Fig. 2), various coefficient values (e.g., CCs, SRCs, PCCs, RCCs, SRRCs, PRCCs) can also be used to assess variable importance (Table II). In Table II and other similar tables in this presentation, CCs and RCCs are calculated between individual pairs of variables, and SRCs and SRRCs are calculated with all sampled variables included in the regression model (i.e., x_1, x_2, x_3 in this example; see Eq. (17)). In the complete absence of correlations between the sampled variable values, corresponding CCs and SRCs would be the same and so would corresponding RCCs and SRRCs. As indicated by the similarity of the values for CCs and SRCs and also for RCCs and SRRCs, there is little correlation between the sampled variables. Further, because an exact linear model is under consideration, PCCs and PRCCs are equal to one. Thus, for a linear model, PCCs and PRCCs provide no information on the importance of individual variables. Because of the linearity of the model, the sample of size $nLHS$ = 1000 gives results almost identical to those in Table II for $nLHS$ = 100.

An alternative summary of the SRCs and SRRCs in Table II is to present the sensitivity results in the form of a stepwise regression analysis (Table III). Then, variable importance is indicated by the order in which the variables entered the regression model, the sizes and signs of the SRCs or SRRCs, and the changes in R^2 values as additional variables are added to the regression model. Because a linear model is under consideration, the stepwise process ultimately produces a regression model with an R^2 value of 1.00. However, the last variable added to the regression model (i.e., x_1) has little effect and only raises the R^2 value from 0.99 to 1.00. The regression coefficients do not provide information on variable importance (i.e., they are all 1.00); rather, it is the SRCs that provide an indication of variable importance. The results in Table III are for raw data; use of rank-transformed data produces similar results.

When a linear relationship exists between a predicted variable and multiple input variables, stepwise regression analysis provides more information on variable importance than simply examining CCs. First, the changes in R^2 values as additional variables are added to the regression model provides an indication of how much uncertainty can be accounted for by each variable. For example, the R^2 values produced with the addition of each variable to the regression model in Table III are 0.89, 0.99 and 1.00, respectively. Thus, the last variable selected (i.e., x_1) only changes the R^2 value from 0.99 to 1.00. Second, the F-test for the sequential addition of variables to the regression model is more sensitive than the test for the significance of a single CC. For example, the p-value obtained with *nLHS* = 100 for the CC associated with x_1 is 0.5091 (Table I); in contrast, the p-value for the entry of x_1 into the regression model that already contains x_3 and x_2 is less than 10^{-4}.

The second linear test problem (Model 3, Ref. [55]) is defined by

$$f(\mathbf{x}) = \sum_{i=1}^{22} c_i \left(x_i - 1/2\right), \ \mathbf{x} = [x_1, x_2, \ldots, x_{22}], \tag{18}$$

with $x_i : U(0, 1)$ and $c_i = (i - 11)^2$ for $i = 1, 2, \ldots, 22$.

Latin hypercube and random sampling produce estimates of similar stability for the CDF for $f(\mathbf{x})$ (Fig. 3). This is different from the first linear function, where Latin hypercube sampling produced more stable estimates (Fig. 1). This stability probability results from the fact that the model can be written as

$$f(\mathbf{x}) = c_{22}\left(x_{22} - 1/2\right) + \sum_{i=1}^{10} c_i \left[\left(x_i - 1/2\right) + \left(x_{22-i} - 1/2\right)\right], \tag{19}$$

which tends to smooth the effects of the random sampling owing to each c_i for $i = 1, 2, \ldots, 10$ being multiplied by the sum of two random values.

For the LHS of size *nLHS* = 100, CCs and RCCs identify the same variables as affecting f (i.e., x_{22}, x_{21}, x_1, x_{20}, x_3, x_2, x_{19}, x_{18}, x_4 with p-values less than 0.05) (Table IV). Similar identifications are also made for CMNs and CLs; in contrast, CMDs and SI fail to identify some of the variables identified by CCs and RCCs. For the LHS of size *nLHS* = 1000, all tests identify more variables as affecting f (Table IV). Further, there is more agreement between the tests on the most important variables (i.e., smallest p-values). However, a number of variables are not identified as having an effect on f by any of the tests (e.g., x_7, x_{15}, x_{14}, x_8, x_9, x_{12}, x_{13}, x_{11}, x_{10} have p-values greater than 0.05 for most tests).

Given that a linear model is under consideration, stepwise regression provides a more informative summary of variable effects than the coefficients in Table IV (Table V). In particular, the stepwise regression analysis with *nLHS* = 100 identifies the effects of all 21 variables that influence the evaluation of f (i.e., all variables except x_{11}, which has a coefficient of zero). The results for $nLHS$ = 1000 (not shown) are essentially identical with those for $nLHS$ = 100; thus, no improvement in the results of the stepwise regression analysis is obtained by increasing the sample size. Thus, the tests of significance used with the stepwise regression analysis are more effective in identifying the effects of individual variables than the tests used in conjunction with Table IV. In particular, the stepwise regression in Table V correctly identifies the effects of all variables influencing f with a sample of size $nLHS$ = 100; the test based on CCs in Table IV does not identify the effects of all variables with a sample of size $nLHS$ = 1000 (i.e., some variables have p-values greater than 0.1).

The cumulative R^2 values with the entry of each variable into the regression model are shown in Table V. The increase in the R^2 value with the entry of a variable shows the fraction of the total uncertainty that can be accounted for by that variable in a linear regression model (e.g., x_{21} accounts for a fraction 0.36279 − 0.20948 = 0.15331 of the total uncertainty). As indicated by the incremental R^2 values, no single variable dominates the uncertainty in f.

For perspective, scatterplots for the first two variables selected in the stepwise process (i.e., x_{22}, x_{21}) are shown in Fig. 4. Although the patterns are discernible, they are not strong, which is consistent with the incremental R^2 values of 0.20948 and 0.15311 associated with x_{22} and x_{21}.

Both regression coefficients and SRCs are given in Table V. The SRCs are a better measure of variable importance because they incorporate the effects of a variable's distribution and also remove the effects of units. Except for the effects of correlations within a sample, CCs and SRCs are the same; thus, the CCs between the x_i and $f(\mathbf{x})$ are also available from Table V. For example, Fig. IV contains scatterplots with associated CCs of approximately 0.46052 for x_{22} and 0.38038 for x_{21}.

4. Monotonic Test Problems

The first monotonic test problem (Model 4, Ref. [55]) is defined by

$$f(\mathbf{x}) = x_1 + x_2^4, \quad \mathbf{x} = [x_1, x_2], \tag{20}$$

with $x_i : U(0, 1)$ for i = 1, 2 (Model 4a), $x_i : U(0, 3)$ for i = 1, 2 (Model 4b), or $x_i : U(0, 5)$ for i = 1, 2 (Model 4c). Thus, f is the same in Models 4a, 4b, and 4c, but the distributions assigned to the x_i change. In the following, Models 4a and 4c will be considered as this incorporates the two extremes in the behavior of f.

Latin hypercube sampling produces more stable estimates of the CDFs for Models 4a and 4c than is the case for random sampling (Fig. 5). This stability is particularly noticeable for Model 4c, where the value of $f(\mathbf{x})$ is dominated by a strong nonlinear relationship involving x_2; in this problem, the stratification associated with Latin hypercube sampling produces CDF estimates that are much more stable than those obtained with random sampling.

Sensitivity analysis for Model 4c is not very interesting due to the dominance of $f(\mathbf{x})$ by x_2 (Fig. 6), with the result that all of the sensitivity analysis procedures under consideration identify x_2 as the dominant variable. Sensitivity analysis is more interesting for Model 4a as both x_1 and x_2 affect $f(\mathbf{x})$. Therefore, only sensitivity analysis for Model 4a will be discussed.

All procedures identify x_1 and x_2 as affecting $f(\mathbf{x})$ for Model 4a and the sample of size $nLHS$ = 100 (Table VI). The well-defined effects of x_1 and x_2 can be seen in the corresponding scatterplots (Fig. 7). The patterns are better defined in the scatterplots for $nLHS$ = 1000 but still easily recognizable in the scatterplots for $nLHS$ = 100.

For perspective, various coefficients (i.e., CCs, SRCs, PCCs, RCCs, SRRCs, PRCCs) involving x_1, x_2 and $f(\mathbf{x})$ are presented in Table VII. As should be the case, CCs and SRCs are similar in size and PCCs are larger than CCs and SRCs; similar patterns also hold for RCCs, SRRCs and PRCCs. In this example, the coefficients calculated with raw (i.e., untransformed) data have values that are similar to those of the corresponding coefficients calculated with rank-transformed data. Thus, the problem is not as nonlinear over the distributions of x_1 and x_2 as might be suggested by the definition of f in Eq. (20), which is consistent with the linear trends appearing in the scatterplots in Fig. 7. The use of samples of size $nLHS$ = 100 and $nLHS$ = 1000 produce similar coefficient values. Thus, the behavior of the function is being adequately captured with $nLHS$ = 100, and little is gained by using a large sample size (although the scatterplots are more visually appealing for $nLHS$ = 1000).

The sensitivity results for Model 4a can also be summarized as the outcome of a stepwise regression analysis (Table VIII). As already observed in conjunction with Table VII, x_1 is identified as having a stronger effect on the uncertainty in $f(\mathbf{x})$ than x_2, and analyses with raw (i.e., untransformed) data and rank-transformed data produce similar results. Use of the sample of size $nLHS$ = 1000 produces little improvement in the regression analyses, with R^2 values for the final regression model changing from 0.88580 and 0.87966 with raw and rank-transformed data with $nLHS$ = 100 to 0.88356 and 0.88482 for $nLHS$ = 1000 (regressions not shown). Thus, as previously noted, increasing the sample size in this example does not improve the results of the sensitivity analysis.

The use of regression analysis with rank-transformed data rather than raw data produced no improvement in the resultant regression model for Model 4a (Table VIII). However, the potential exists for considerable improvement when the dependent variable is a nonlinear but monotonic function of the independent variable(s). For example, a nonlinear but monotonic relationship exists between x_2 and $f(\mathbf{x})$ for Model 4c (Fig. 6). In the analysis of this model, a regression with rank-transformed data relating $f(\mathbf{x})$ to x_2 with $nLHS$ = 100 produces a regression model with an R^2 value of 0.97574; the corresponding regression with raw data produces a regression model with an R^2 value of 0.75003.

The rank transformation is self-standardizing in the sense that RRCs and SRRCs are essentially equal (Table VIII), with strict equality holding in the absence of equal (i.e., tied) variable values and approximate equality holding when equal variable values result in the use of average ranks for the equal values.

The second monotonic test problem (Model 5, Ref. [55]) is defined by

$$f(\mathbf{x}) = \exp\left(\sum_{i=1}^{6} b_i x_i\right) - \prod_{i=1}^{6}\left(e^{b_i} - 1\right)\Big/ b_i, \quad \mathbf{x} = [x_1, x_2, \ldots, x_6], \tag{21}$$

with $b_1 = 1.5$, $b_2 = b_3 = \cdots = b_6 = 0.9$ and $x_i : U(0, 1)$ for $i = 1, 2, \ldots, 6$.

Latin hypercube sampling produces more stable estimates of the CDF for $f(\mathbf{x})$ than does random sampling (Fig. 8). However, the distribution has a long tail to the right, and both sampling procedures show considerable variation across replicates in the largest observed value for $f(\mathbf{x})$. Thus, if accurate estimates of the upper quantiles of the CDF were required, then it would be necessary to use a large sample size or possibly switch to an importance sampling procedure. For functions that are as inexpensive to evaluate as f, it would be wasteful to invest the effort to design an importance sampling procedure. However, as the cost of evaluating the function (i.e., model) increases, at some point use of importance sampling may become cost effective.

All tests (i.e., CCs, RCCs, CMNs, CLs, CMDs, SI) identify x_1 as the most important variable for $nLHS = 100$ (Table IX); further, CCs and RCCs identify effects for all six x_i. Given the definition of f, x_1 is the most important variable with respect to the uncertainty in $f(\mathbf{x})$, and $x_2, x_3, \ldots, x_6$ have equal-sized effects on this uncertainty. For $nLHS = 1000$, all tests identify effects for all six x_i.

The coefficients (i.e., CCs, SRCs, PCCs, RCCs, SRRCs, PRCCs) involving the x_i and $f(\mathbf{x})$ are presented in Table X. The largest coefficients involve x_1; $x_2, x_3, \ldots, x_6$ have similar-sized coefficients; CCs and SRCs are essentially equal, as is the case for RCCs and SRRCs; PCCs and PRCCs are larger than the corresponding CCs and RCCs, respectively; and all coefficients are positive, which is consistent with the use of the x_i in the definition of $f(\mathbf{x})$. Samples of size $nLHS = 100$ and $nLHS = 1000$ produce similar coefficient estimates.

The scatterplots for x_1 and x_2 show discernible, but not particularly strong, patterns (Fig. IX). As should be the case given the definition of $f(\mathbf{x})$ and the distributions assigned to the x_i, the scatterplots for x_1 show somewhat stronger patterns than the scatterplots for x_2. The scatterplots for x_3, x_4, x_5, x_6 are similar to those for x_2.

The sensitivity results for Model 5 can also be presented as stepwise regression analyses with raw and rank-transformed data (Table XI). The regression analyses with both raw and rank-transformed data identify the effects associated with all six x_i's. Further, the regression analyses with rank-transformed data produce models with higher R^2 values than the regression analyses with raw data (i.e., 0.94119 versus 0.74993 for $nLHS = 100$ and 0.96285 versus

0.80030 for *nLHS* = 1000). There is little difference in the regression results obtained with *nLHS* = 100 and *nLHS* = 1000 (not shown).

5. Nonmonotonic Test Problems

The first nonmonotonic test problem (Model 7, Ref. [55]) is defined by

$$\begin{aligned} f(\mathbf{x}) &= \prod_{i=1}^{8} g_i(x_i), \quad \mathbf{x} = [x_1, x_2, \ldots, x_8] \\ &= \prod_{i=1}^{8} \frac{|4x_i - 2| + a_i}{1 + a_i} \end{aligned} \tag{22}$$

with $a_1 = 0$, $a_2 = 1$, $a_3 = 4.5$, $a_4 = 9$, $a_5 = a_6 = a_7 = a_8 = 99$, and x_i: $U(0, 1)$ for $i = 1, 2, \ldots, 8$.

Latin hypercube sampling produces estimates of the CDF for $f(\mathbf{x})$ that are more stable than those produced by random sampling (Fig. 10).

Tests based on CCs and RCCs fail to identify any of the x_i as affecting $f(\mathbf{x})$ for *nLHS* = 100 and also for *nLHS* = 1000 (Table XII). In contrast, tests based on CMNs, CLs, CMDs and SI identify significant effects for x_1 and x_2 for both *nLHS* = 100 and *nLHS* = 1000, with the exception that the SI test does not identify x_2 for *nLHS* = 100. In addition, smaller effects are indicated for x_3 (CMN, CL, CMD) and x_4 (CMN, CL, CMD, SI) for *nLHS* = 1000. Tests based on CCs and RCCs fail to identify the effects of x_1 and x_2 on $f(\mathbf{x})$ because these effects are both nonlinear and nonmonotonic (Fig. 11). In contrast, such effects are readily identified by CMNs, CLs, CMDs and SI. All the coefficients involving $f(\mathbf{x})$ and the x_i's (i.e., CCs, SRCs, PCCs, RCCs, SRRCs, PRCCs) are essentially zero; similarly, the regression analyses with raw and rank-transformed data produce no meaningful results.

The second nonmonotonic test problem (Model 8, Ref. [55]) is defined by

$$f(\mathbf{x}) = h(x_2) \sum_{i=0}^{[x_2/2]} c_i(x_2)_i(x_1, x_2), \tag{23}$$

where h, c_i and g_i are defined by

$$h(x_2) = 2^{-x_2}, \quad c_i(x_2) = (-1)^i \binom{x_2}{i}\binom{2x_2 - 2i}{x_2}, \quad g_i(x_1, x_2) = x_1^{x_2 - 2i},$$

and x_1: $U(-1, 1)$, x_2: $DU(5)$, [~] designates the greatest integer function, and x: $DU(n)$ indicates that x has a uniform distribution over the integers $j = 1, 2, \ldots, n$.

Latin hypercube sampling and random sampling produce estimates of the CDF for $f(\mathbf{x})$ that exhibit similar stability (Fig. 12). This behavior is in contrast to the other examples, where Latin hypercube sampling tends to produce more stable CDF estimates than random sampling.

For $nLHS = 100$, tests based on CMNs and SI identify an effect for x_1 (i.e., p-values < 0.05) (Table XIII). The test based on CLs with a p-value of 0.0723 also suggests an effect for x_1. None of the remaining tests (i.e., CCs, RCCs, CMDs) indicates an effect for x_1. The test based on SI with a p-value of 0.0698 suggests a possible effect for x_2; none of the other tests have p-values that suggest an effect for x_2. For $nLHS = 1000$, all tests indicate an effect for x_1, and the test based on SI also indicates an effect for x_2.

This example has complex patterns involving x_1 and x_2 (Fig. 13). These patterns partially emerge for $nLHS = 100$ and are readily apparent for $nLHS = 1000$. Of the tests under consideration, the test based on SI is most effective in identifying these patterns. Due to the complexity of the relations involving x_1, x_2 and $f(\mathbf{x})$, none of the previously considered coefficients (i.e., CCs, SRCs, PCCs, RCCs, SRRCs, PRCCs) have values that provide any useful insights on these relationships. Similarly, stepwise regression analyses with raw and rank-transformed data fail to provide any useful insights.

The third nonmonotonic test problem (Model 9, Ref. [55]) is defined by

$$f(\mathbf{x}) = \sin x_1 + A \sin^2 x_2 + B x_3^4 \sin x_1, \quad \mathbf{x} = [x_1, x_2, x_3], \tag{24}$$

with $A = 7$, $B = 0.1$, and x_i: $U(-\pi, \pi)$ for $i = 1, 2, 3$.

For this example, the CDF estimates obtained with Latin hypercube sampling are more stable than those obtained with random sampling (Fig. 14).

In sensitivity analyses with $nLHS = 100$, all tests identify x_1 as affecting $f(\mathbf{x})$ (Table XIV). In addition, the CMNs, CLs, CMDs and SI tests also identify an effect for x_2. None of the tests identifies an effect for x_3. For $nLHS = 1000$, all tests indicate an effect for x_1, and tests based on CMNs, CLs, CMDs and SI indicate an effect for x_2. In contrast, CCs and RCCs fail to indicate an effect for x_2. Further, the test based on SI also identifies an effect for x_3.

Examination of scatterplots clearly shows that x_1, x_2 and x_3 have readily discernible influences on $f(\mathbf{x})$ (Fig. 15). The tests based on CCs and RCCs are completely missing the nonlinear and nonmonotonic patterns induced in $f(\mathbf{x})$ by x_2 and x_3. Tests based on CCs and RCCs are able to identify a slight increasing pattern in the relationship between x_1 and $f(\mathbf{x})$; but this is only part of the patterns appearing in Fig. 15. Tests based on CMNs, CLs and CMDs identify the pattern associated with x_2 but fail to identify the pattern associated with x_3 that tends to produce similar means and medians across the entire range of x_3. In contrast, this pattern was detected by the test for SI with $nLHS = 1000$.

Due to the lack of strong linear or monotonic relationships between x_1, x_2, x_3 and $f(\mathbf{x})$, individual coefficients (i.e., CCs, SRCs, PCCs, RCCs, SRRCs, PRCCs) are close to zero and provide little useful information to help in determining the effects of x_1, x_2 and x_3 on $f(\mathbf{x})$. For the same reasons, stepwise regression analysis with raw or rank-transferred data is not very informative.

6. Discussion

This presentation uses relatively simple test problems to illustrate sampling-based procedures for uncertainty and sensitivity analysis. Such simplicity helps in understanding the techniques in use but is not typical of real problems. Many examples of real, and hence more complex, analyses using sampling-based procedures are available.[1-13, 70-83]

The complexity of many real analysis problems is increased by the presence of both stochastic (i.e., aleatory) uncertainty and subjective (i.e., epistemic) uncertainty.[24-31] Stochastic uncertainty arises because the system under study can behave in many different ways and thus is a property of the system. Subjective uncertainty arises from an inability to specify the exact value of a quantity that is assumed to have a fixed value within a particular analysis and thus is a property of the analysts carrying out the study. The distinction between stochastic and subjective uncertainty can be traced back to the beginnings of the formal development of probability theory in the late sixteen hundreds.[84-86]

The test problems in this presentation are assumed to involve subjective uncertainty. The analysis of problems that involve both stochastic and subjective uncertainty requires careful planning and implementation. Often, event trees and fault trees are used to represent the effects of stochastic uncertainty, and sampling-based procedures of the type illustrated in this presentation are used to represent the effects of subjective uncertainty. Many examples of analyses involving both stochastic and subjective uncertainty exist, including analyses related to reactor safety,[87-90] radioactive waste disposal,[91-93] environmental risk assessment,[94-100] and petroleum exploration.[101]

Many approaches are available for uncertainty and sensitivity analysis, including differential analysis,[102-115] response surface methodology,[116-126] the Fourier amplitude sensitivity test (FAST),[127-131] variance decomposition,[132-141] and fast probability integration.[142-148] Differential analysis involves approximating a model with a Taylor series and then using variance propagation formulas to obtain uncertainty and sensitivity analysis results. Response surface methodology is based on using classical experimental designs to select points for use in developing a response surface replacement for a model; this replacement model is then used in subsequent uncertainty and sensitivity analyses based on variance propagation and Monte Carlo simulation. The FAST is based on using techniques from Fourier analysis to decompose the variance of a model prediction into the components due to individual model inputs and is closely related to the other indicated variance decomposition procedures. With the FAST and other variance decomposition procedures, uncertainty and sensitivity analysis results are based on the variance of model predictions and the contribution of individual model inputs to this variance. Fast probability integration is an uncertainty analysis technique used to estimate the tails of the uncertainty distributions for model predictions.

Although many approaches to uncertainty and sensitivity analysis exist, a sampling-based approach is usually a suitable, and quite often the best, approach for various combinations of the following reasons: (i) conceptual simplicity and ease of implementation (e.g., unlike other methods, there are no requirements for model differentiation, complex experimental designs and associated response surface construction, or high dimensional integrations), (ii) dense stratification over the range of each sampled variable, especially when Latin hypercube sampling is used, (iii) direct provision of uncertainty analysis results without the use of surrogate models as approximations to the original model (e.g., Taylor series or response surfaces), (iv) availability of a variety of sensitivity analysis procedures, and (v) effectiveness as a model verification procedure (i.e., exploration of the mapping from uncertain inputs to model results provides a powerful tool for the identification of errors in model construction and analysis implementation).

A concern often expressed about sampling-based uncertainty and sensitivity analyses is that the number of required model evaluations will make the cost of the analysis prohibitive. In practice, this is usually not the case. In most analyses, a sample size of considerably less than 1000 is sufficient to obtain useful uncertainty and sensitivity analysis results. This is certainly the case for the test problems considered in this presentation and has been demonstrated in a number of real analyses.[69, 92, 149, 150]

Several points need to be kept in mind when considering the computational cost associated with sampling-based uncertainty and sensitivity analyses. First, high quantiles of distributions representing subjective uncertainty are typically not needed, and in addition, are usually not meaningful. Specifically, a general idea of the uncertainty range in a model's predictions is important to have but to know something such as the 0.999 quantile of the distribution is usually not very useful. Further, in most analyses, the resolution at which the subjective uncertainty in a model's inputs can be assessed does not justify ascribing any real meaning to very low or very high quantiles of resulting uncertainty distributions. Second, the belief that estimates for extreme quantiles is needed often comes from confusing stochastic and subjective uncertainty. In many analyses, stochastic uncertainty deals with rare events (e.g., unlikely accidents) that really could happen. In such analyses, the estimation of extreme quantiles is important and is typically carried out with an importance sampling procedure defined and implemented through the use of event trees. Third, the uncertainty in a given analysis result is usually dominated by the uncertainty in only a few inputs. As a result, a large sample size is not needed for an effective uncertainty and sensitivity analysis. The preceding does not have to be true but is typically true in practice. Fourth, implementation of the other previously indicated uncertainty and sensitivity analysis techniques can often require as many or more model evaluations as a sampling-based analysis, and thus, have equal or greater computational cost. Finally, in most analyses, the cost of the human time to develop the model, characterize the uncertainty in model inputs, and carry out the analysis is much greater than the cost of the required model evaluations.

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Figure Captions

Fig. 1. Stability of estimated CDF for linear test problem with Model 1 (see Eq. (17).

Fig. 2. Scatterplots for linear test problem with Model 1 (see Eq. (17)).

Fig. 3. Stability of estimated CDF for linear test problem with Model 3 (see Eq. (18)).

Fig. 4. Scatterplots for linear test problem with Model 3 (see Eq. (18)).

Fig. 5. Stability of estimated CDFs for monotonic test problem with Models 4a and 4c (see Eq. (20)).

Fig. 6. Scatterplot with *nLHS* = 100 for monotonic test problem with Model 4c (see Eq. (20)).

Fig. 7. Scatterplots for monotonic test problem with Model 4a (see Eq. (20)).

Fig. 8. Stability of estimated CDF for monotonic test problem with Model 5 (see Eq. (21)).

Fig. 9. Scatterplots for monotonic test problem with Model 5 (see Eq. (21)).

Fig. 10. Stability of estimated CDF for nonmonotonic test problem with Model 7 (see Eq. (22)).

Fig. 11. Scatterplots for nonmonotonic test problem with Model 7 (see Eq. (22)).

Fig. 12. Stability of estimated CDF for nonmonotonic test problem with Model 8 (see Eq. (23)).

Fig. 13. Scatterplots for nonmonotonic test problem with Model 8 (see Eq. (23).

Fig. 14. Stability of estimated CDF for nonmonotonic test problem with Model 9 (see Eq. (24)).

Fig. 15. Scatterplots for nonmonotonic test problem with Model 9 (see Eq. (24)).

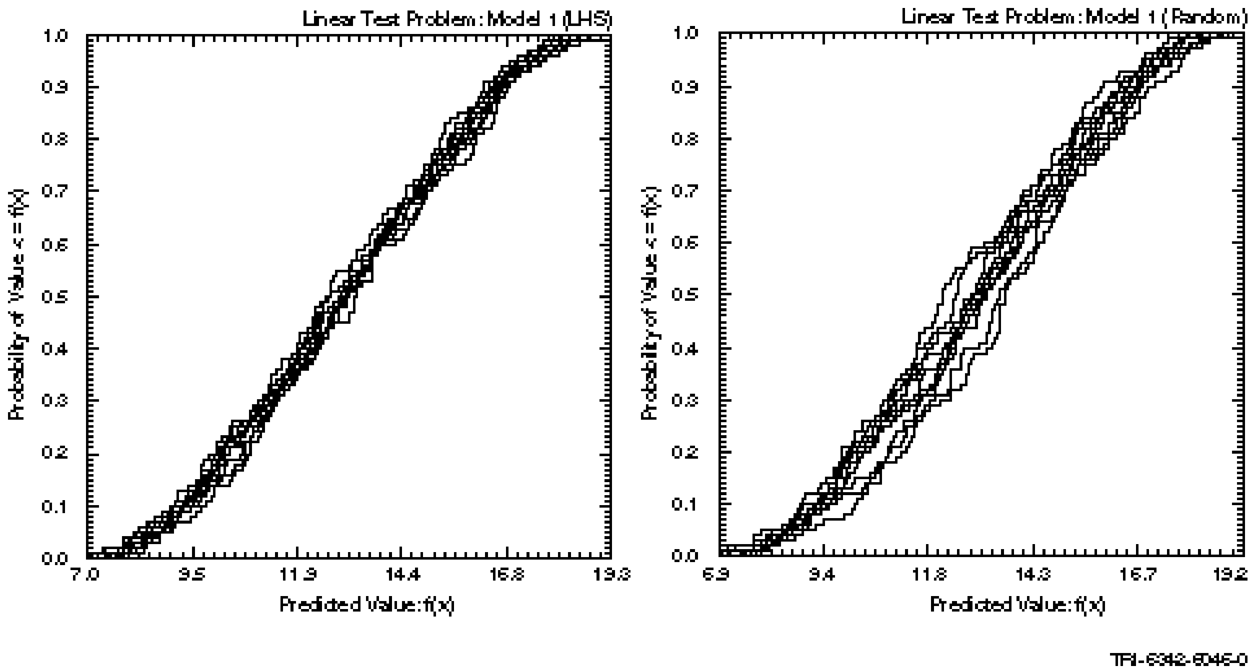

Fig. 1. Stability of estimated CDF for linear test problem with Model 1 (see Eq. (17).

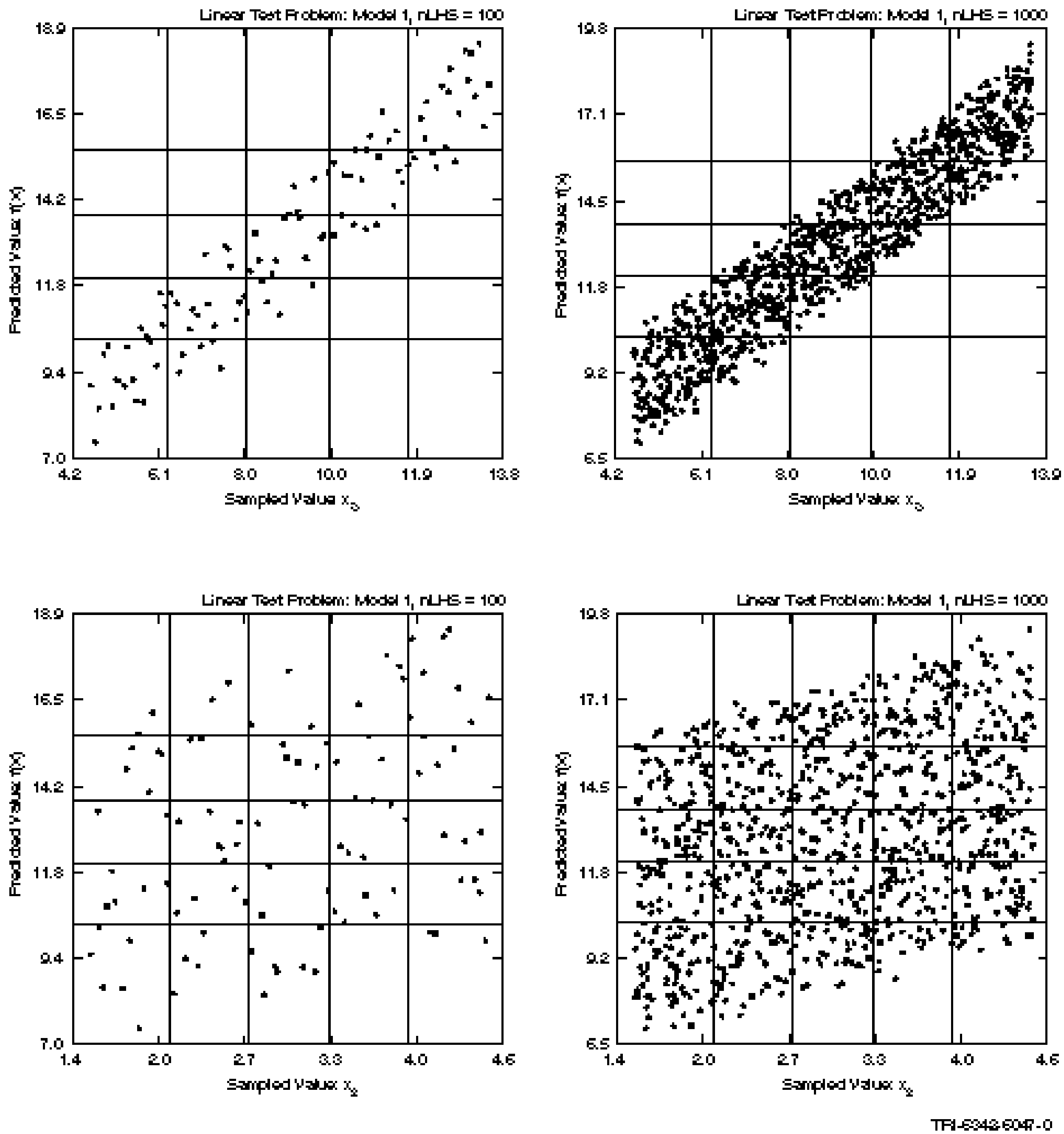

Fig. 2. Scatterplots for linear test problem with Model 1 (see Eq. (17)).

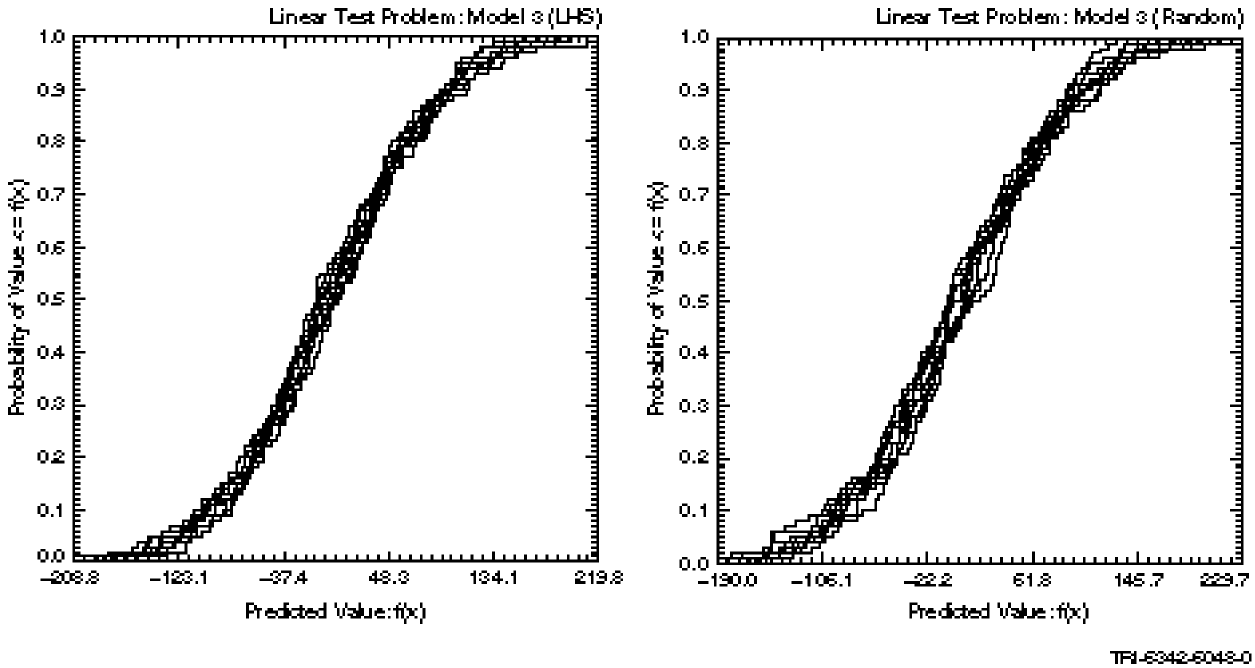

Fig. 3. Stability of estimated CDF for linear test problem with Model 3 (see Eq. (18)).

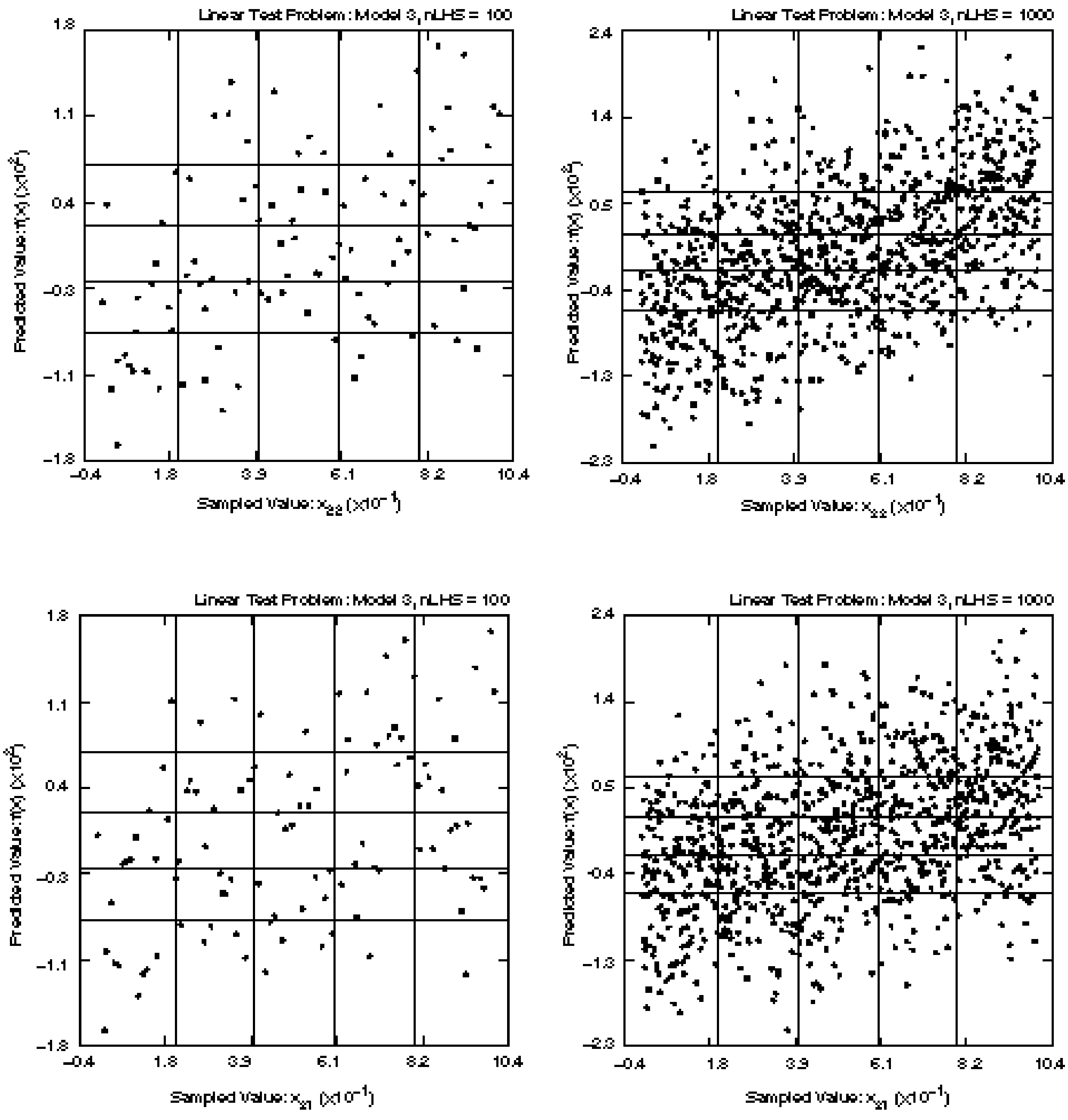

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Fig. 4. Scatterplots for linear test problem with Model 3 (see Eq. (18)).

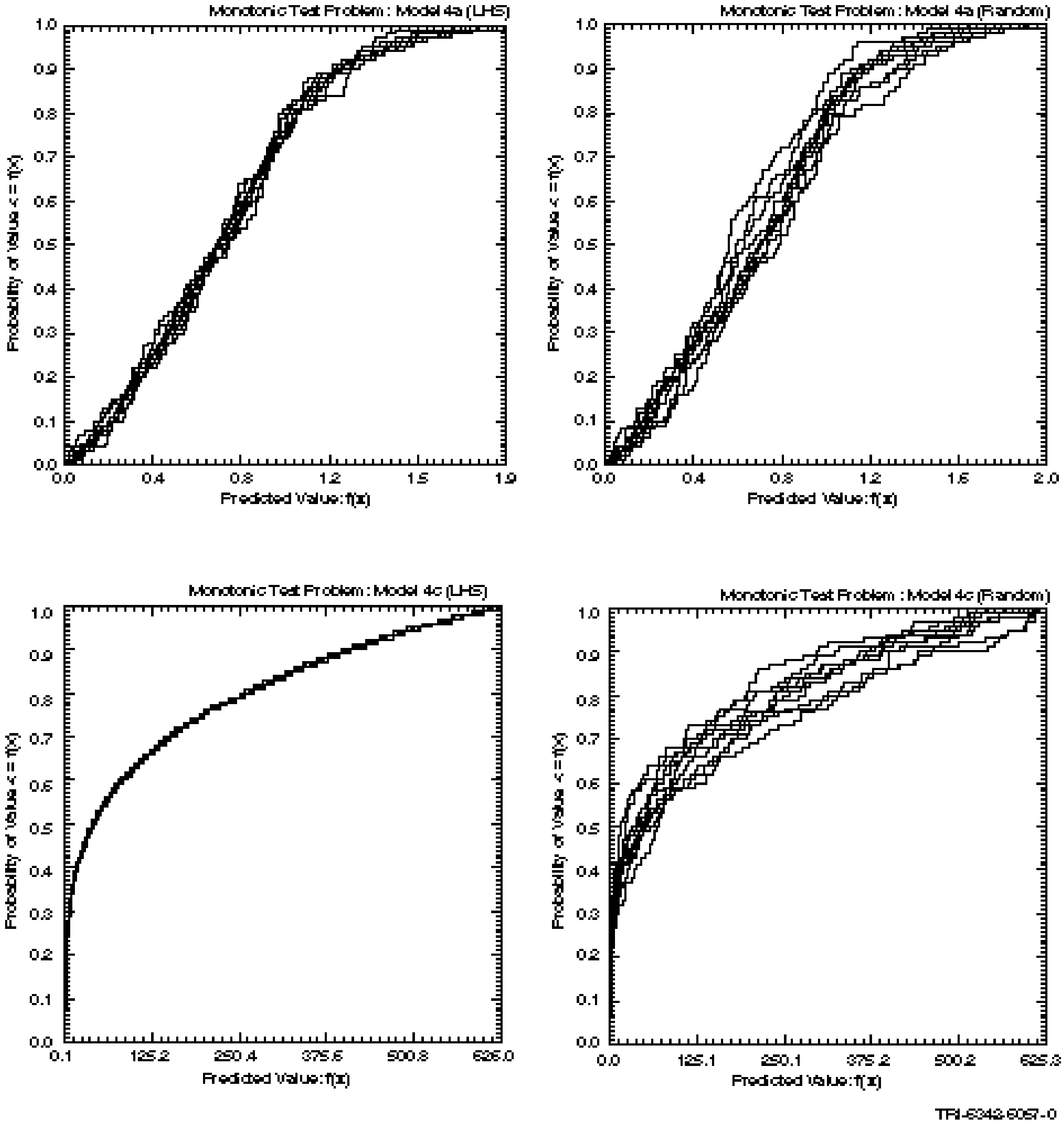

Fig. 5. Stability of estimated CDFs for monotonic test problem with Models 4a and 4c (see Eq. (20)).

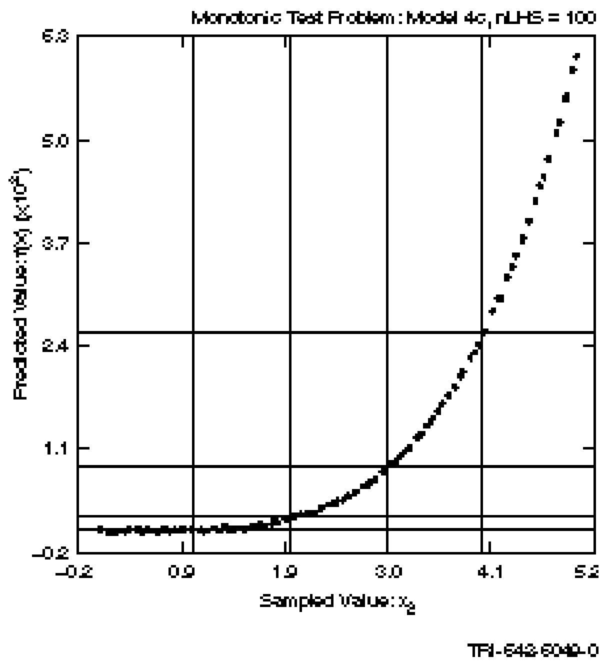

Fig. 6. Scatterplot with *nLHS* = 100 for monotonic test problem with Model 4c (see Eq. (20)).

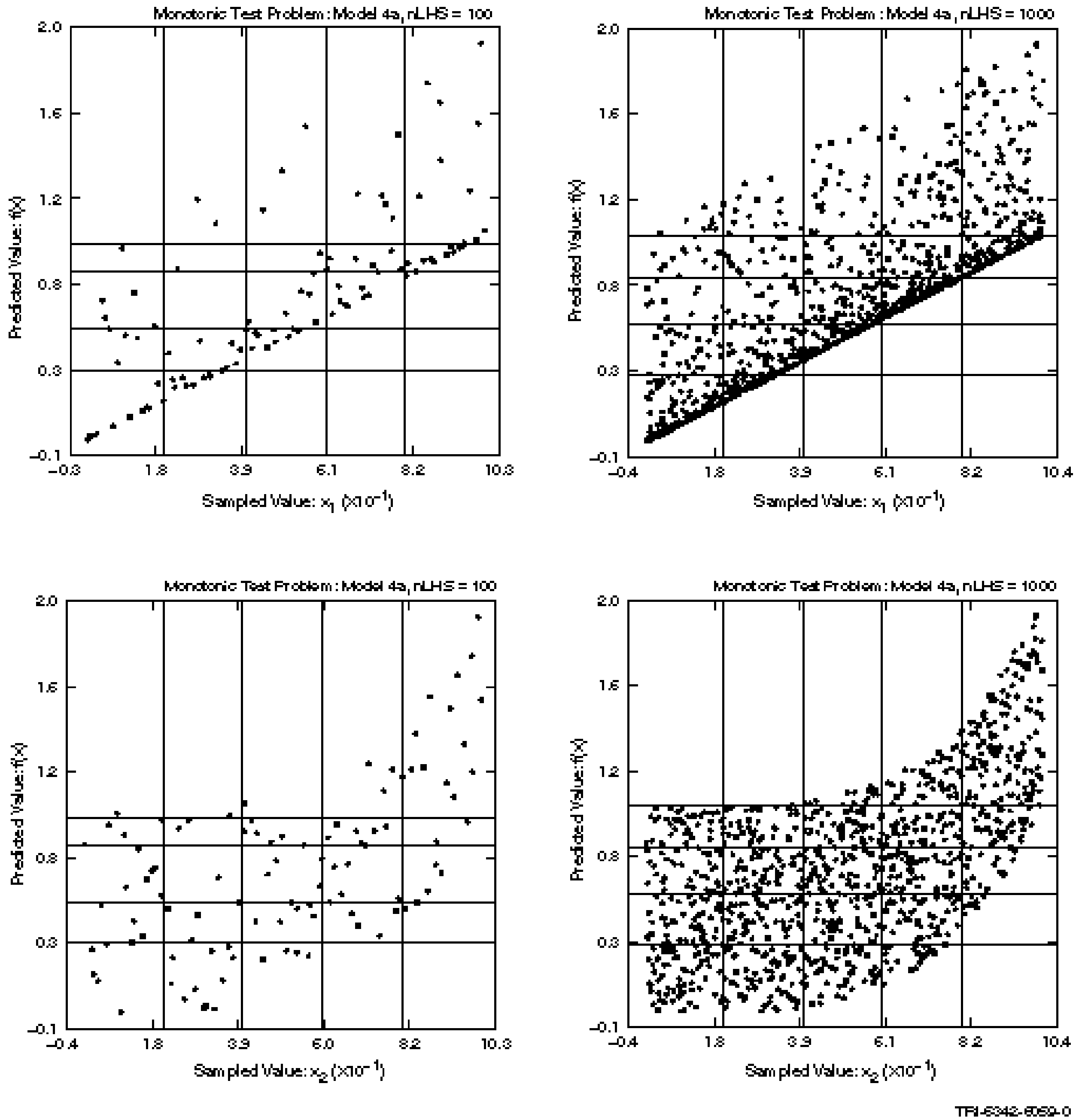

Fig. 7. Scatterplots for monotonic test problem with Model 4a (see Eq. (20)).

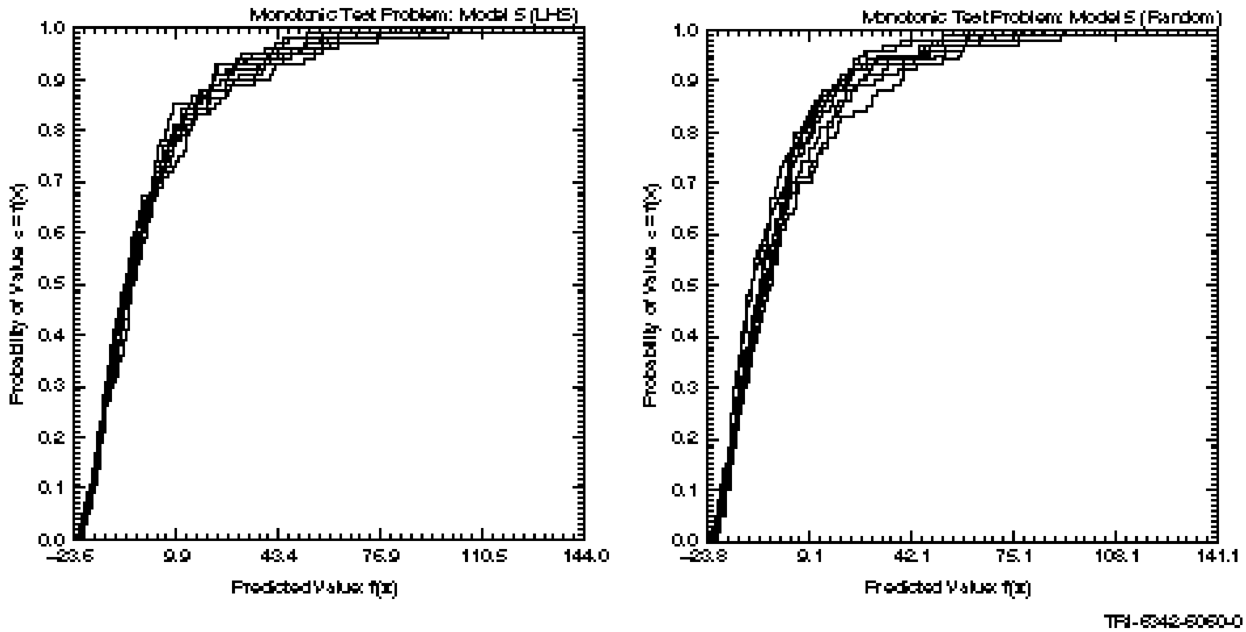

Fig. 8. Stability of estimated CDF for monotonic test problem with Model 5 (see Eq. (21)).

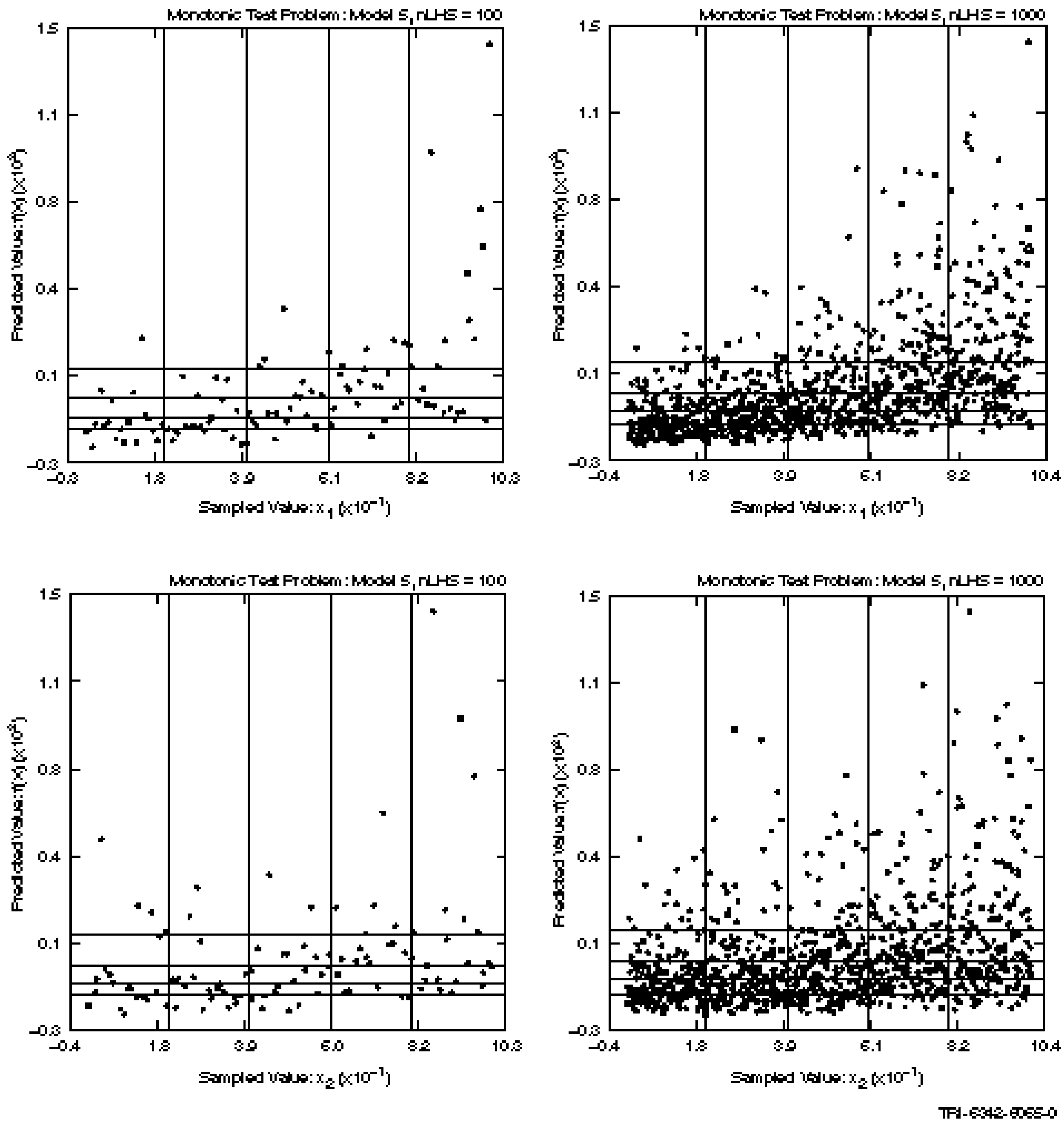

Fig. 9. Scatterplots for monotonic test problem with Model 5 (see Eq. (21)).

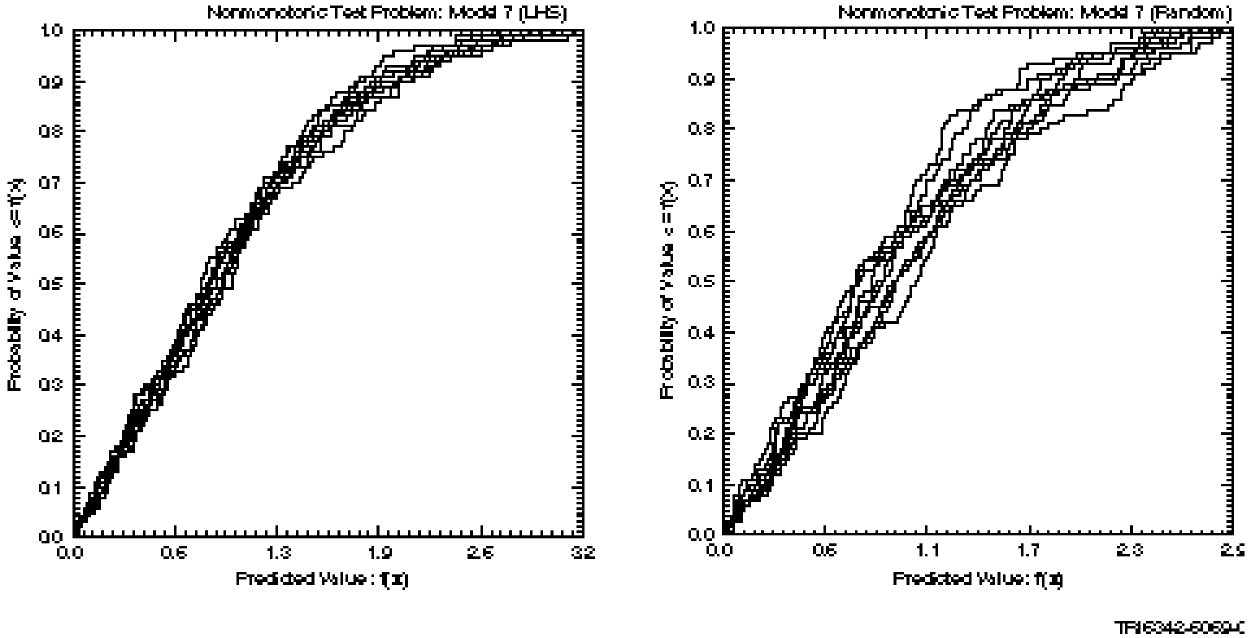

Fig. 10. Stability of estimated CDF for nonmonotonic test problem with Model 7 (see Eq. (22)).

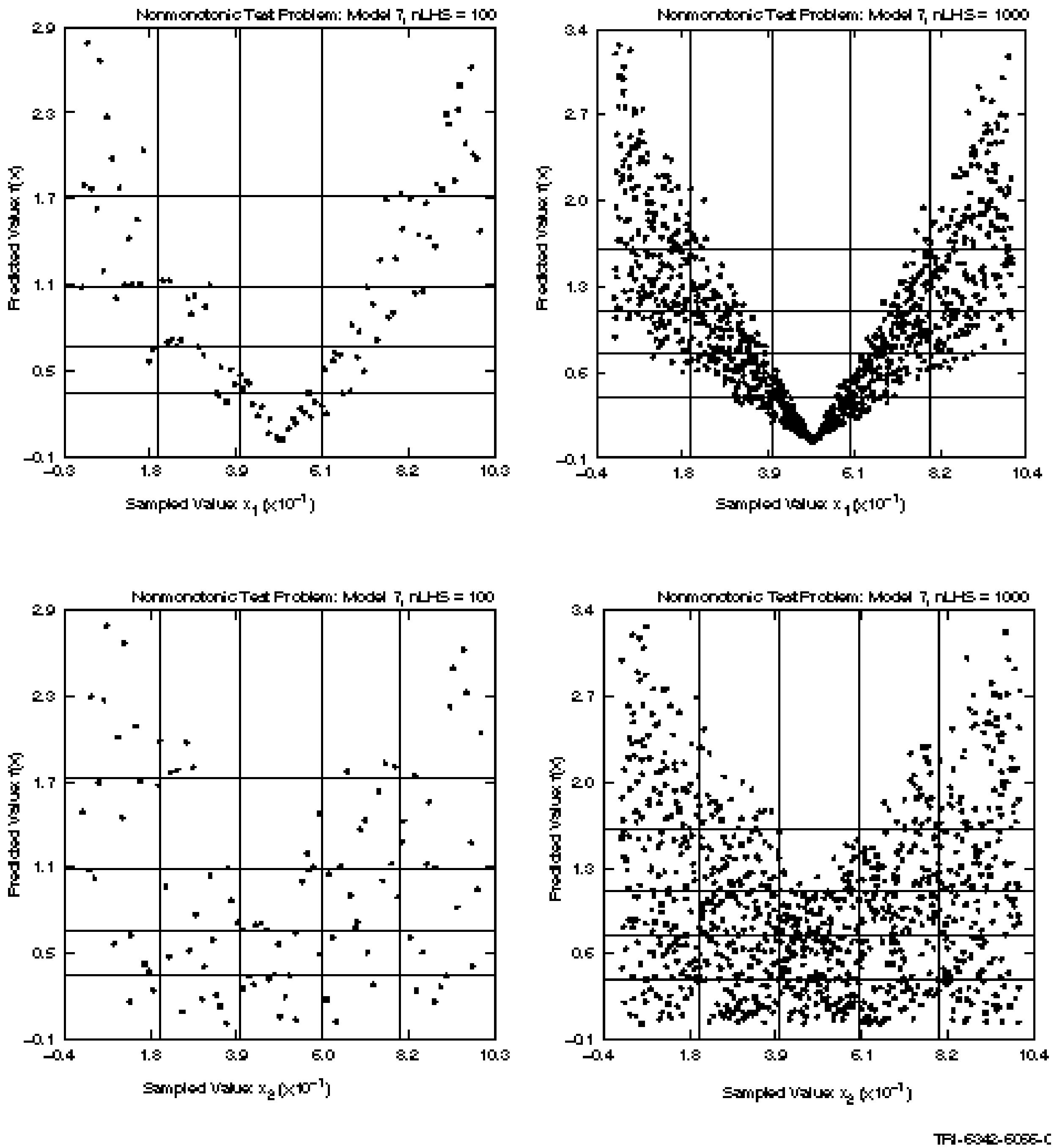

Fig. 11. Scatterplots for nonmonotonic test problem with Model 7 (see Eq. (22)).

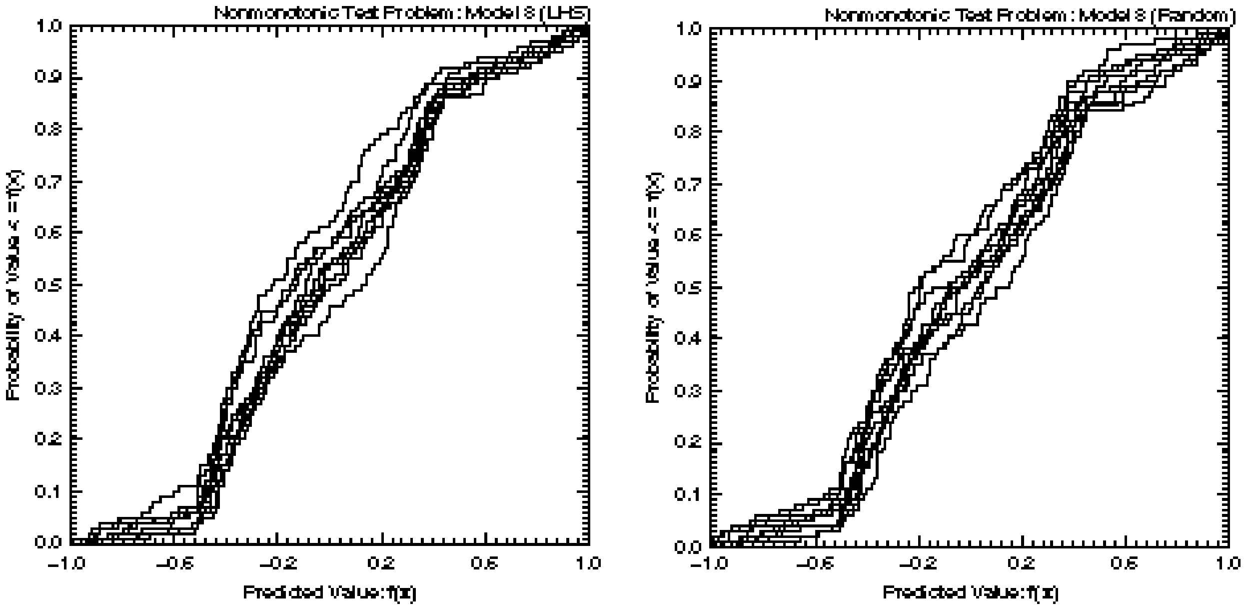

Fig. 12. Stability of estimated CDF for nonmonotonic test problem with Model 8 (see Eq. (23)).

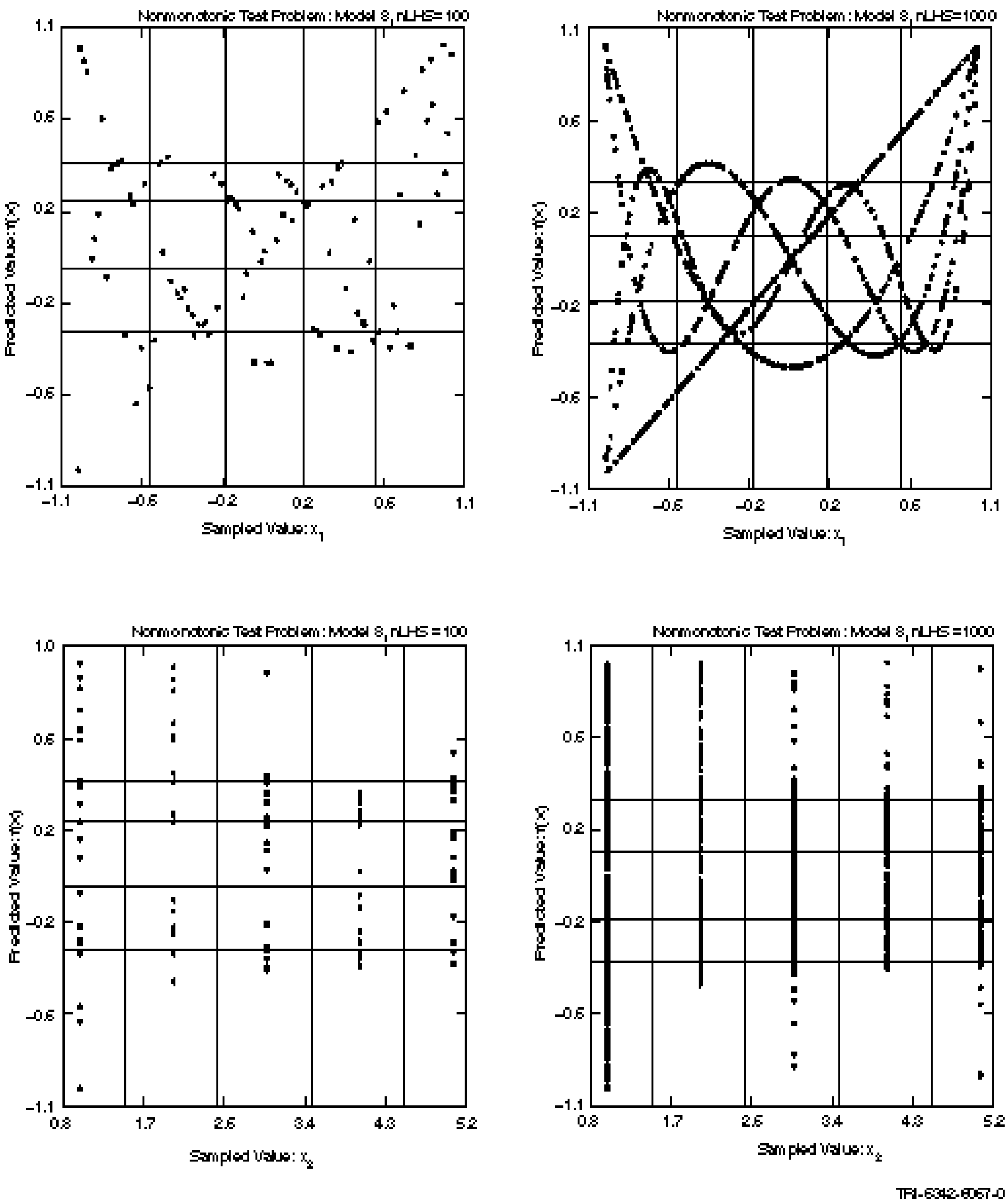

Fig. 13. Scatterplots for nonmonotonic test problem with Model 8 (see Eq. (23)).

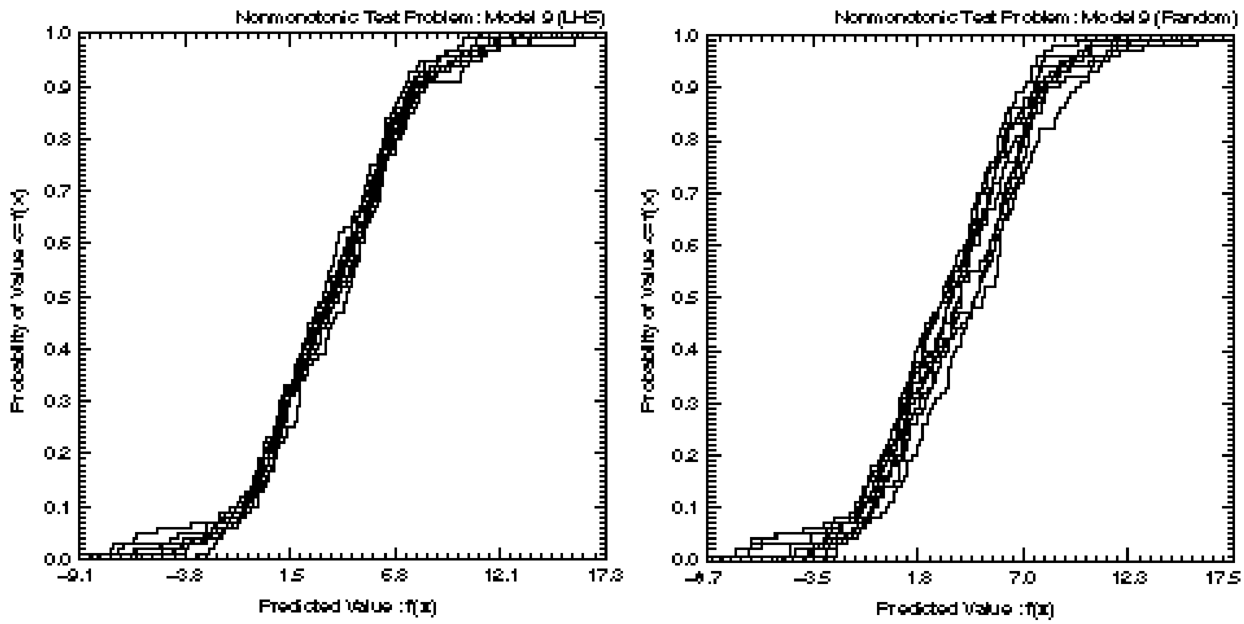

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Fig. 14. Stability of estimated CDF for nonmonotonic test problem with Model 9 (see Eq. (24)).

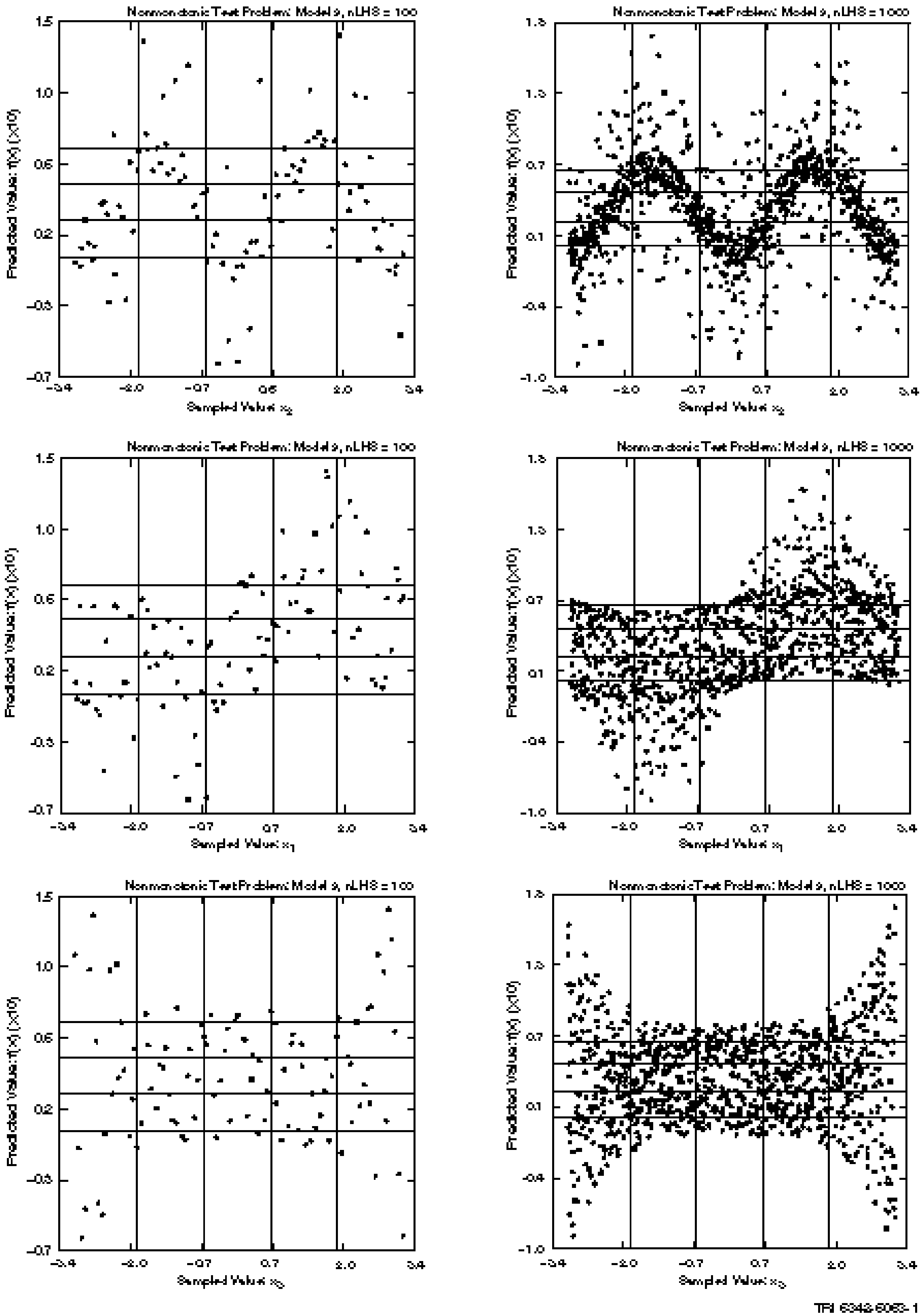

Fig. 15. Scatterplots for nonmonotonic test problem with Model 9 (see Eq. (24)).

Table I. Sensitivity Results Based on CCs, RCCs, CMNs, CLs, CMDs and SI for Linear Test Problem with Model 1 (see Eq. (17))

Sample Size: $nLHS$ = 100

Variable Name[a]	CC[b]		RCC[c]		CMN[d]		CL[e]		CMD[f]		SI[g]	
	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val
x_3	1.0	0.0000	1.0	0.0000	1.0	0.0000	1.0	0.0000	1.0	0.0000	1.0	0.0000
x_2	2.0	0.0015	2.0	0.0027	2.0	0.0502	2.0	0.0779	2.0	0.5249	2.0	0.2954
x_1	3.0	0.5091	3.0	0.5694	3.0	0.7528	3.0	0.7089	3.0	0.7358	3.0	0.8392

Sample Size: $nLHS$ = 1000

Variable Name	CC		RCC		CMN		CL		CMD		SI	
	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val
x_3	1.0	0.0000	1.0	0.0000	1.0	0.0000	1.0	0.0000	1.0	0.0000	1.0	0.0000
x_2	2.0	0.0000	2.0	0.0000	2.0	0.0000	2.0	0.0000	2.0	0.0000	2.0	0.0000
x_1	3.0	0.0007	3.0	0.0017	3.0	0.0155	3.0	0.0313	3.0	0.4748	3.0	0.1164

[a] Variables ordered by p-values for CCs

[b] Ranks and p-values for CCs

[c] Ranks and p-values for RCCs

[d] Ranks and p-values for CMNs test with 1×5 grid (i.e., division of x values into 5 intervals of equal probability and no division of y values).

[e] Ranks and p-values for CLs (Kruskal-Wallis) test with 1×5 grid (i.e., division of x values into 5 intervals of equal probability and no division of y values).

[f] Ranks and p-values for CMDs text with 2×5 grid (i.e., division of x values into 5 intervals of equal probability and division of y values into 2 intervals defined by the median of the y values).

[g] Ranks and p-values for SI test with 5×5 grid (i.e., division of both x and y values into 5 intervals of equal probability).

Table II. Sensitivity Results Based on Coefficients (i.e., CCs, SRCs, PCCs, RCCs, SRRCs, PRCCs) and Sample Size $nLHS$ = 100 for Linear Test Problem with Model 1 (see Eq. (17))

Variable Name[a]	CC[b]		SRC[b]		PCC[b]		RCC[b]		SRRC[b]		PRCC[b]	
	Rank	Value	Rank	Value	Rank	Value	Rank	Value	Rank	Value	Rank	Value
x_3	1	0.9439	1	0.9459	2	1.000	1	0.9466	1	0.9482	2	1.000
x_2	2	0.3175	2	0.3156	2	1.000	2	0.3018	2	0.2987	2	1.000
x_1	3	0.0660	3	0.1054	2	1.000	3	0.0572	3	0.0976	2	1.000

[a] Variables ordered by p-values for CCs.

[b] Ranks and values for CCs, SRCs, PCCs, RCCs, SRRCs and PRCCs as indicated.

Table III. Sensitivity Results Based on Stepwise Regression Analysis with Raw (i.e., Untransformed) Data and Sample Size $nLHS$ = 100 for Linear Test Problem with Model 1 (see Eq. (17))

Variable[a]	R^{2}[b]	RC[c]	SRC[d]	p-Value[e]
x_3	0.89098	1.0000E+00	9.4588E–01	0.0000E+00
x_2	0.98891	1.0000E+00	3.1558E–01	0.0000E+00
x_1	1.00000	1.0000E+00	1.0541E–01	0.0000E+00

[a] Variables in order of entry into regression model.
[b] Cumulative R^2 value with entry of each variable into regression model.
[c] Regression coefficients (RCs) in final regression model.
[d] Standardized regression coefficients (SRCs) in final regression model.
[e] For variable in row (i.e., x_i), p- or α-value for addition of x_i to regression model containing remaining variables.

Table IV. Sensitivity Results Based on CCs, RCCs, CMNs, CLs, CMDs and SI for Linear Test Problem with Model 3 (see Eq. (18))[a]

Sample Size: *nLHS* = 100

Variable Name	CC		RCC		CMN		CL		CMD		SI	
	Rank	*p*-Val	Rank	*p*-Val	Rank	*p*-Val	Rank	*p*-Val	Rank	*p*-Val	Rank	*p*-Val
x_{22}	1.0	0.0000	1.0	0.0000	1.0	0.0001	1.0	0.0002	1.5	0.0009	2.0	0.0208
x_{21}	2.0	0.0001	2.0	0.0002	2.0	0.0004	4.0	0.0018	9.0	0.1074	1.0	0.0086
x_1	3.0	0.0002	3.0	0.0003	5.0	0.0024	5.0	0.0043	7.0	0.0289	8.5	0.1137
x_{20}	4.0	0.0003	4.0	0.0005	7.0	0.0070	7.0	0.0131	11.0	0.2311	5.0	0.0615
x_3	5.0	0.0015	5.0	0.0016	6.0	0.0064	6.0	0.0086	4.0	0.0103	3.0	0.0239
x_2	6.0	0.0028	7.0	0.0037	3.0	0.0006	2.0	0.0012	3.0	0.0051	6.0	0.0791
x_{19}	7.0	0.0037	6.0	0.0025	10.0	0.0699	10.0	0.0445	5.0	0.0123	10.5	0.1785
x_{18}	8.0	0.0238	8.0	0.0197	8.0	0.0318	8.0	0.0294	6.0	0.0146	12.0	0.2202
x_4	9.0	0.0444	9.0	0.0289	9.0	0.0399	9.0	0.0295	10.0	0.1991	8.5	0.1137
x_{17}	10.0	0.1095	10.0	0.1135	12.0	0.1476	12.0	0.1515	15.5	0.4060	10.5	0.1785
x_{16}	11.0	0.1379	11.0	0.1154	20.0	0.7358	18.0	0.6699	13.5	0.3546	4.0	0.0316
x_5	12.0	0.2668	12.0	0.3349	4.0	0.0012	3.0	0.0016	1.5	0.0009	7.0	0.1010
x_6	13.0	0.4991	13.0	0.4195	18.0	0.6822	19.0	0.6835	18.0	0.5249	14.0	0.4186
x_9	14.0	0.5118	18.0	0.6595	17.0	0.3711	16.0	0.4258	18.0	0.5249	15.5	0.4884
x_7	15.0	0.5261	16.0	0.5194	19.0	0.7351	20.0	0.7596	15.5	0.4060	18.0	0.5987
x_8	16.0	0.5368	15.0	0.5006	14.0	0.3476	17.0	0.4307	18.0	0.5249	13.0	0.3239
x_{12}	17.0	0.5487	14.0	0.4632	16.0	0.3656	14.0	0.3570	20.0	0.7358	18.0	0.5987
x_{13}	18.0	0.7118	17.0	0.6491	13.0	0.2392	13.0	0.2676	13.5	0.3546	15.5	0.4884
x_{14}	19.0	0.8221	21.0	0.9223	21.0	0.9511	21.0	0.9651	22.0	0.9825	18.0	0.5987
x_{15}	20.0	0.8317	20.0	0.7924	22.0	0.9922	22.0	0.9929	21.0	0.9384	20.0	0.6359
x_{11}	21.0	0.8909	19.0	0.7495	11.0	0.0716	11.0	0.1020	8.0	0.0404	21.0	0.7440
x_{10}	22.0	0.9217	22.0	0.9840	15.0	0.3507	15.0	0.3963	12.0	0.3084	22.0	0.7776

Sample Size: *nLHS* = 1000

Variable Name	CC		RCC		CMN		CL		CMD		SI	
	Rank	*p*-Val	Rank	*p*-Val	Rank	*p*-Val	Rank	*p*-Val	Rank	*p*-Val	Rank	*p*-Val
x_{22}	1.0	0.0000	1.0	0.0000	1.0	0.0000	1.0	0.0000	1.0	0.0000	1.0	0.0000
x_{21}	2.0	0.0000	2.0	0.0000	2.0	0.0000	3.0	0.0000	2.0	0.0000	3.0	0.0000
x_1	3.0	0.0000	3.0	0.0000	3.0	0.0000	2.0	0.0000	3.0	0.0000	2.0	0.0000
x_2	4.0	0.0000	4.0	0.0000	4.0	0.0000	4.0	0.0000	6.0	0.0000	4.0	0.0000
x_{20}	5.0	0.0000	5.0	0.0000	5.0	0.0000	5.0	0.0000	4.0	0.0000	5.0	0.0000
x_3	6.0	0.0000	6.0	0.0000	6.0	0.0000	6.0	0.0000	5.0	0.0000	6.0	0.0000
x_{19}	7.0	0.0000	7.0	0.0000	7.0	0.0000	7.0	0.0000	7.0	0.0000	7.0	0.0000
x_{18}	8.0	0.0000	8.0	0.0000	8.0	0.0000	8.0	0.0000	8.0	0.0000	9.0	0.0001
x_4	9.0	0.0000	9.0	0.0000	9.0	0.0000	9.0	0.0000	9.0	0.0000	10.0	0.0003
x_5	10.0	0.0000	11.0	0.0000	10.0	0.0000	10.0	0.0000	10.0	0.0002	8.0	0.0000
x_{17}	11.0	0.0000	10.0	0.0000	11.0	0.0002	11.0	0.0002	12.0	0.0040	12.0	0.0121
x_{16}	12.0	0.0011	12.0	0.0002	12.0	0.0124	12.0	0.0035	11.0	0.0004	13.0	0.1164
x_6	13.0	0.0018	13.0	0.0014	13.0	0.0252	13.0	0.0212	13.0	0.0206	11.0	0.0019
x_7	14.0	0.0637	14.0	0.0776	14.0	0.0267	14.0	0.0697	16.0	0.1538	14.0	0.1164
x_{15}	15.0	0.0959	15.0	0.0892	22.0	0.6771	20.0	0.5827	21.0	0.7431	21.0	0.6691
x_{14}	16.0	0.2579	18.0	0.3909	18.0	0.4664	19.0	0.5414	18.0	0.4809	15.0	0.1843
x_8	17.0	0.3165	16.0	0.2949	19.0	0.5583	18.0	0.4750	14.0	0.0425	17.0	0.2509
x_9	18.0	0.3907	19.0	0.4178	20.0	0.5701	21.0	0.7113	22.0	0.9437	18.0	0.2899

x_{12}	19.0	0.4606	17.0	0.3616	16.0	0.3026	16.0	0.1976	15.0	0.1402	16.0	0.1944
x_{13}	20.0	0.4625	21.0	0.5867	15.0	0.1438	15.0	0.1348	17.0	0.2792	19.0	0.2954
x_{11}	21.0	0.6626	20.0	0.5806	17.0	0.3446	17.0	0.3143	19.0	0.4932	20.0	0.4530
x_{10}	22.0	0.7892	22.0	0.7117	21.0	0.6605	22.0	0.7753	20.0	0.5512	22.0	0.9950

[a] Table structure same as in Table I.

Table V. Sensitivity Results Based on Stepwise Regression Analysis with Raw (i.e., Untransformed) Data and Sample Size *nLHS* = 100 for Linear Test Problem with Model 3 (see Eq. (18))[a]

Variable	R^2	RC	SRC	p-Value
x_{22}	0.20948	1.2100E+02	4.6052E−01	2.7828E−08[b]
x_{21}	0.36279	1.0000E+02	3.8038E−01	2.7828E−08
x_1	0.50981	1.0000E+02	3.8141E−01	2.7828E−08
x_{20}	0.63339	8.1000E+01	3.0763E−01	2.7828E−08
x_2	0.73563	8.1000E+01	3.0830E−01	2.7828E−08
x_3	0.80541	6.4000E+01	2.4338E−01	2.7828E−08
x_{19}	0.86382	6.4000E+01	2.4317E−01	2.7828E−08
x_{18}	0.90285	4.9000E+01	1.8642E−01	2.7828E−08
x_4	0.93449	4.9000E+01	1.8614E−01	2.7828E−08
x_5	0.95728	3.6000E+01	1.3677E−01	2.7828E−08
x_{17}	0.97297	3.6000E+01	1.3665E−01	2.7828E−08
x_6	0.98146	2.5000E+01	9.5070E−02	2.7828E−08
x_{16}	0.98978	2.5000E+01	9.5121E−02	2.7828E−08
x_{15}	0.99340	1.6000E+01	6.0789E−02	2.7828E−08
x_7	0.99710	1.6000E+01	6.0905E−02	2.7828E−08
x_8	0.99833	9.0000E+00	3.4256E−02	2.7828E−08
x_{14}	0.99950	9.0000E+00	3.4263E−02	2.7828E−08
x_9	0.99974	4.0000E+00	1.5206E−02	2.7828E−08
x_{13}	0.99997	4.0000E+00	1.5225E−02	2.7828E−08
x_{10}	0.99999	9.9999E−01	3.8041E−03	2.7828E−08
x_{12}	1.00000	1.0000E+00	3.8018E−03	2.7828E−08
x_{11}	1.00000	−3.0113E−05	−1.1426E−07	2.6792E−01

[a] Table structure same as in Table III.
[b] Identical values result from lack of resolution in algorithm used in the calculation of very small p-values.

Table VI. Sensitivity Results Based on CCs, RCCs, CMNs, CLs, CMDs and SI for Monotonic Test Problem with Model 4a (see Eq. (20)) and *nLHS* = 100[a]

Variable Name	CC		RCC		CMN		CL		CMD		SI	
	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val
x_1	1.0	0.0000	1.0	0.0000	1.0	0.0000	1.0	0.0000	1.0	0.0000	1.0	0.0000
x_2	2.0	0.0000	2.0	0.0000	2.0	0.0000	2.0	0.0000	2.0	0.0004	2.0	0.0000

[a] Table structure same as in Table I.

Table VII. Sensitivity Results Based on Coefficients (i.e., CCs, SRCs, PCCs, RCCs, SRRCs, PRCCs) for Monotonic Test Problem with Model 4a (see Eq. (20))

Sample Size: *nLHS* = 100							
Variable Name[a]	CC[b]			SRC[c]		PCC[c]	
	p-Value	Rank	Value	Rank	Value	Rank	Value
x_1	0.0000	1.0	0.7367	1.0	0.7401	1.0	0.9097
x_2	0.0000	2.0	0.5814	2.0	0.5857	2.0	0.8662
Variable Name[d]	RCC[e]			SRRC[f]		PRCC[f]	
	p-Value	Rank	Value	Rank	Value	Rank	Value
x_1	0.0000	1.0	0.7688	1.0	0.7723	1.0	0.9122
x_2	0.0000	2.0	0.5322	2.0	0.5373	2.0	0.8401
Sample Size: *nLHS* = 1000							
Variable Name	CC			SRC		PCC	
	p-Value	Rank	Value	Rank	Value	Rank	Value
x_1	0.0000	1.5	0.7310	1.0	0.7263	1.0	0.9051
x_2	0.0000	1.5	0.5967	2.0	0.5910	2.0	0.8660
Variable Name	RCC			SRRC		PRCC	
	p-Value	Rank	Value	Rank	Value	Rank	Value
x_1	0.0000	1.5	0.7531	1.0	0.7489	1.0	0.9108
x_2	0.0000	1.5	0.5692	2.0	0.5637	2.0	0.8567

[a] Variables ordered by *p*-values for CCs.

[b] *p*-values, ranks and values for CCs.

[c] Ranks and values for SRCs and PCCs as indicated.

[d] Variables ordered by *p*-values for RCCs.

[e] *p*-values, ranks and values for RCCs.

[f] Ranks and values for SRRCs and PRCCs as indicated.

Table VIII. Sensitivity Results Based on Stepwise Regression Analysis for Monotonic Test Problem with Model 4a (see Eq. (20)) and Sample Size *nLHS* = 100[a]

Raw Data				
Variable	R^2	RC	SRC	*p*-Value
x_1	0.54273	1.0070E+00	7.4014E–01	2.7828E–08
x_2	0.88580	7.9861E–01	5.8573E–01	2.7828E–08
Rank-Transformed Data				
Variable	R^2	RRC[b]	SRRC[c]	*p*-Value
x_1	0.59099	7.7229E–01	7.7229E–01	2.7828E–08
x_2	0.87966	5.3728E–01	5.3728E–01	2.7828E–08

[a] Table structure same as in Table III.
[b] Rank regression coefficient (RRC).
[c] Standardized rank regression coefficient (SRRC).

Table IX. Sensitivity Results Based on CCs, RCCs, CMNs, CLs, CMDs and SI for Monotonic Test Problem with Model 5 (see Eq. (21))[a]

Sample Size: $nLHS$ = 100												
Variable Name	CC		RCC		CMN		CL		CMD		SI	
	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val
x_1	1.0	0.0000	1.0	0.0000	1.0	0.0000	1.0	0.0000	1.0	0.0000	1.0	0.0000
x_4	2.0	0.0005	5.0	0.0009	3.0	0.0161	4.0	0.0095	6.0	0.1468	5.0	0.2436
x_5	3.0	0.0007	6.0	0.0029	2.0	0.0006	2.0	0.0011	3.0	0.0342	2.0	0.0156
x_2	4.0	0.0041	4.0	0.0007	4.0	0.0211	5.0	0.0098	2.0	0.0087	3.0	0.0180
x_6	5.0	0.0051	3.0	0.0004	6.0	0.0840	6.0	0.0184	4.0	0.0477	6.0	0.4530
x_3	6.0	0.0052	2.0	0.0003	5.0	0.0464	3.0	0.0070	5.0	0.0780	4.0	0.0540
Variable Name	CC		RCC		CMN		CL		CMD		SI	
	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val
x_1	1.0	0.0000	1.0	0.0000	1.0	0.0000	1.0	0.0000	1.0	0.0000	1.0	0.0000
x_5	2.0	0.0000	6.0	0.0000	3.0	0.0000	4.0	0.0000	5.0	0.0000	5.0	0.0000
x_2	3.0	0.0000	2.0	0.0000	2.0	0.0000	2.0	0.0000	2.0	0.0000	2.0	0.0000
x_4	4.0	0.0000	4.0	0.0000	5.0	0.0000	3.0	0.0000	3.0	0.0000	4.0	0.0000
x_3	5.0	0.0000	3.0	0.0000	4.0	0.0000	6.0	0.0000	6.0	0.0000	6.0	0.0000
x_6	6.0	0.0000	5.0	0.0000	6.0	0.0000	5.0	0.0000	4.0	0.0000	3.0	0.0000

[a] Table structure same as in Table I.

Table X. Sensitivity Results Based on Coefficients (i.e., CCs, SRCs, PCCs, RCCs, SRRCs, PRCCs) for Monotonic Test Problem with Model 5 (see Eq. (21))[a]

Sample Size: *nLHS* = 100							
Variable Name	CC			SRC		PCC	
	p-Value	Rank	Value	Rank	Value	Rank	Value
x_1	0.0000	1.0	0.5078	1.0	0.5223	1.0	0.7221
x_4	0.0005	2.0	0.3459	3.0	0.3446	3.0	0.5673
x_5	0.0007	3.0	0.3371	2.0	0.3509	2.0	0.5739
x_2	0.0041	4.0	0.2868	5.0	0.2952	5.0	0.5080
x_6	0.0051	5.0	0.2803	6.0	0.2837	6.0	0.4929
x_3	0.0052	6.0	0.2793	4.0	0.2973	4.0	0.5108
Variable Name	RCC			SRRC		PRCC	
	p-Value	Rank	Value	Rank	Value	Rank	Value
x_1	0.0000	1.0	0.5852	1.0	0.6013	1.0	0.9273
x_3	0.0003	2.0	0.3596	2.0	0.3763	2.0	0.8404
x_6	0.0004	3.0	0.3591	3.0	0.3669	3.0	0.8339
x_2	0.0007	4.0	0.3405	4.0	0.3456	4.0	0.8183
x_4	0.0009	5.0	0.3334	5.0	0.3317	5.0	0.8071
x_5	0.0029	6.0	0.2992	6.0	0.3142	6.0	0.7912
Sample Size: *nLHS* = 1000							
Variable Name	CC			SRC		PCC	
	p-Value	Rank	Value	Rank	Value	Rank	Value
x_1	0.0000	1.0	0.5259	1.0	0.5217	1.0	0.7594
x_5	0.0000	2.0	0.3412	2.0	0.3367	2.0	0.6017
x_2	0.0000	3.0	0.3297	4.0	0.3241	4.0	0.5871
x_4	0.0000	4.0	0.3275	3.0	0.3251	3.0	0.5882
x_3	0.0000	5.0	0.3274	5.0	0.3220	5.0	0.5846
x_6	0.0000	6.0	0.3032	6.0	0.3044	6.0	0.5629
Variable Name	RCC			SRRC		PRCC	
	p-Value	Rank	Value	Rank	Value	Rank	Value
x_1	0.0000	1.0	0.5960	1.0	0.5917	1.0	0.9508
x_2	0.0000	2.0	0.3624	2.0	0.3558	2.0	0.8792
x_3	0.0000	3.0	0.3553	4.0	0.3486	3.0	0.8751
x_4	0.0000	4.0	0.3484	5.0	0.3462	5.0	0.8736
x_6	0.0000	5.0	0.3467	3.0	0.3486	4.0	0.8751
x_5	0.0000	6.0	0.3431	6.0	0.3380	6.0	0.8687

[a] Table structure same as in Table VII.

Table XI. Sensitivity Results Based on Stepwise Regression Analysis for Monotonic Test Problem with Model 5 (see Eq. (21)) and Sample Size $nLHS$ = 100[a]

Raw Data				
Variable	R^2	RC	SRC	p-Value
x_1	0.25787	4.4071E+01	5.2230E−01	2.7828E−08
x_5	0.37674	2.9727E+01	3.5091E−01	2.9036E−08
x_4	0.49249	2.9194E+01	3.4459E−01	2.9872E−08
x_2	0.58539	2.4960E+01	2.9519E−01	1.7598E−07
x_3	0.66967	2.5164E+01	2.9734E−01	1.5130E−07
x_6	0.74993	2.4008E+01	2.8369E−01	4.1674E−07
Rank-Transformed Data				
Variable	R^2	RRC	SRRC	p-Value
x_1	0.34245	6.0130E−01	6.0130E−01	2.7828E−08
x_6	0.48424	3.6689E−01	3.6689E−01	2.7828E−08
x_3	0.62262	3.7628E−01	3.7628E−01	2.7828E−08
x_2	0.73162	3.4561E−01	3.4561E−01	2.7828E−08
x_4	0.84275	3.3165E−01	3.3165E−01	2.7828E−08
x_5	0.94119	3.1419E−01	3.1419E−01	2.7828E−08

[a] Table structure same as in Table VIII.

Table XII. Sensitivity Results Based on CCs, RCCs, CMNs, CLs, CMDs and SI for Nonmonotonic Test Problem with Model 7 (see Eq. (22))[a]

Sample Size $nLHS$ = 100												
Variable Name	CC		RCC		CMN		CL		CMD		SI	
	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val
x_1	1.0	0.1657	1.0	0.2382	1.0	0.0000	1.0	0.0000	1.0	0.0000	1.0	0.0000
x_3	2.0	0.4400	3.0	0.4666	3.0	0.2294	3.0	0.3469	7.0	0.7358	3.0	0.1010
x_8	3.0	0.4518	2.0	0.4090	7.0	0.7298	8.0	0.7661	7.0	0.7358	8.0	0.9489
x_6	4.0	0.4566	6.0	0.5905	5.0	0.6637	6.0	0.7193	4.5	0.5918	7.0	0.8666
x_7	5.0	0.4758	4.0	0.5528	8.0	0.7360	7.0	0.7623	7.0	0.7358	6.0	0.6728
x_5	6.0	0.6796	5.0	0.5860	6.0	0.7179	5.0	0.4218	4.5	0.5918	2.0	0.0698
x_2	7.0	0.7545	8.0	0.9833	2.0	0.0010	2.0	0.0055	2.0	0.0206	4.0	0.1601
x_4	8.0	0.9581	7.0	0.9002	4.0	0.4531	4.0	0.3902	3.0	0.0916	5.0	0.5615
Sample Size $nLHS$ = 1000												
Variable Name	CC		RCC		CMN		CL		CMD		SI	
	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val
x_7	1.0	0.2089	1.0	0.1838	7.0	0.7123	7.0	0.7153	6.0	0.2873	6.0	0.4153
x_6	2.0	0.2644	3.0	0.2813	8.0	0.8882	8.0	0.7586	7.0	0.6411	8.0	0.9394
x_8	3.0	0.2943	2.0	0.2345	6.0	0.6228	6.0	0.4925	8.0	0.7652	7.0	0.6544
x_4	4.0	0.3376	4.0	0.4287	4.0	0.0045	4.0	0.0140	4.0	0.0224	3.0	0.0156
x_2	5.0	0.6614	6.0	0.9430	2.0	0.0000	2.0	0.0000	2.0	0.0000	2.0	0.0000
x_1	6.0	0.7620	8.0	0.9708	1.0	0.0000	1.0	0.0000	1.0	0.0000	1.0	0.0000
x_5	7.0	0.8045	7.0	0.9433	5.0	0.4128	5.0	0.3011	5.0	0.1712	5.0	0.2412

x_3	8.0	0.9197	5.0	0.7315	3.0	0.0001	3.0	0.0034	3.0	0.0220	4.0	0.1178

[a] Table structure same as in Table I.

Table XIII. Sensitivity Results Based on CCs, RCCs, CMNs, CLs, CMDs and SI for Nonmonotonic Test Problem with Model 8 (see Eq. (23))[a]

Sample Size $nLHS$ = 100												
Variable Name	CC		RCC		CMN		CL		CMD		SI	
	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val
x_1	1.0	0.1968	2.0	0.3458	1.0	0.0346	1.0	0.0723	1.0	0.1468	1.0	0.0003
x_2	2.0	0.2412	1.0	0.2722	2.0	0.7078	2.0	0.7449	2.0	0.9384	2.0	0.0698
Sample Size $nLHS$ = 1000												
Variable Name	CC		RCC		CMN		CL		CMD		SI	
	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val
x_1	1.0	0.0000	1.0	0.0000	1.0	0.0000	1.0	0.0000	1.0	0.0000	1.0	0.0000
x_2	2.0	0.6222	2.0	0.0659	2.0	0.9090	2.0	0.2553	2.0	0.1847	2.0	0.0000

[a] Table structure same as in Table I.

Table XIV. Sensitivity Results Based on CCs, RCCs, CMNs, CLs, CMDs and SI for Nonmonotonic Test Problem with Model 9 (see Eq. (24))[a]

Sample Size $nLHS$ = 100												
Variable Name	CC		RCC		CMN		CL		CMD		SI	
	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val
x_1	1.0	0.0000	1.0	0.0000	1.0	0.0000	1.0	0.0000	2.0	0.0001	1.0	0.0000
x_3	2.0	0.5667	2.0	0.6361	3.0	0.6917	3.0	0.5495	3.0	0.9384	3.0	0.0615
x_2	3.0	0.8327	3.0	0.8393	2.0	0.0000	2.0	0.0000	1.0	0.0000	2.0	0.0008
Sample Size $nLHS$ = 1000												
Variable Name	CC		RCC		CMN		CL		CMD		SI	
	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val	Rank	p-Val
x_1	1.0	0.0000	1.0	0.0000	1.5	0.0000	1.5	0.0000	2.0	0.0000	1.5	0.0000
x_3	2.0	0.0162	2.0	0.0187	3.0	0.0438	3.0	0.0347	3.0	0.1446	3.0	0.0000
x_2	3.0	0.9799	3.0	0.9999	1.5	0.0000	1.5	0.0000	1.0	0.0000	1.5	0.0000

[a] Table structure same as in Table I.